2005S M20C Exam 2 Preparation

The second midterm exam is Wednesday May 18, during the class period. *You must bring a blue book to the exam.* Blue books are available from the bookstore. You also need to bring your student ID card or other form of ID (driver's license, passport, etc.) The exam covers material from §14.1-14.8, 15.1 **You may not use a calculator during the exam.**

The following formulæ will be provided on your exam:

$$\nabla f = \langle f_x, f_y \rangle \qquad \mathrm{D}_{\boldsymbol{u}} f = \nabla f \cdot \boldsymbol{u} \qquad D = f_{xx} f_{yy} - f_{yx} f_{xy}$$

Everything else must be committed to memory.

You should be prepared to answer at least the following questions:

1. (§14.1) (this is a basic one) You should understand functions of two or more variables. *e.g.,*: given $f(x, y)$, be able to evaluate $f(1, 2)$, or $f(\pi, 100)$; find the domain of f; sketch a graph of $f(x, y)$, or the contours (level sets) of $f(x, y)$.

1. What is the domain of $f(x, y) = \frac{\sqrt{x-y}}{y}$?
2. What is the domain of $f(x, y) = \frac{xy^3}{x^2+y^2}$?
3. Graph the level sets of $f(x, y) = \frac{1}{4}x^2 + y^2$.
4. Graph the function $f(x, y) = \sqrt{x^2 + y^2}$.

Answer:

1. The points (x, y) with $x \geq y$, and $y \neq 0$.
2. The points (x, y) with $x \neq 0$, and $y \neq 0$. Sometimes written as $\mathbb{R}^2 \setminus \{(0, 0)\}$, that is all the real plane, $\mathbb{R}^2$, without the single point $(0, 0)$.
3. These are ellipses.
4. A paraboloid.

2. (§14.1) Sketch the level sets of $f(x, y)$ for various values of k

1. $f(x, y) = x^2 + y^2$
2. $f(x, y) = 2x - 4y + 7$
3. $f(x, y) = \sqrt{x^2 + y^2}$

Answer:

1. level set of value k is a circle of radius $\sqrt{k}$ centered at the origin.
2. level set of value k is a line with slope $\frac{1}{2}$ and y-intercept of $(7 - k)/4$
3. level set of value k is a circle of radius k centered at the origin.

3. (§14.2) Identify if a function $f(x, y)$ is continuous at a given point (x, y). Identify if the function is continuous for all points in $\mathbb{R}^2$.

1. $f(x, y) = x^2 + 4x^3y^2$ at $(x, y) = (1, 3)$.
2. $f(x, y) = \begin{cases} \frac{x+y}{x^2+y^2} & (x, y) \neq (0, 0) \\ 1 & (x, y) = (0, 0) \end{cases}$ at $(x, y) = (0, 0)$.
3. $f(x, y) = \frac{x^3+y^3}{\sqrt{x+y}}$ at $(x, y) = (2, 1)$.

Answer:

1. This function is continuous for all points in $\mathbb{R}^2$, so it is continuous at $(1, 3)$.
2. The function is *not* continuous at $(0, 0)$, but is continuous at all other points. To show this you should consider the limit of $f(x, y)$ as $(x, y) \to (0, 0)$; this limit is not

defined, thus f is not continuous. However, if you wanted to somehow use continuity of f to *compute* the limit of f as $(x, y) \to (0, 0)$, you would have a "chicken and the egg" problem.

This illustrates the general principle: if you suspect the limit $\lim_{(x,y)\to(a,b)} f(x, y)$ exists, it may be easiest to show the function is continuous at (a, b), then you only need evaluate $f(a, b)$. However, if the function is not continuous at the point, or you think the limit does not exist, this is not the way to go. To show the limit does not exist, it may be better to find two paths approaching (a, b) which give different limits of f.

3. This function is continuous for all points such that $x + y > 0$, thus it is continuous at $(2, 1)$.

4. (§14.2) You should be able to find the limit of a function of two variables, or recognize that the limit does not exist. You need to know what functions are continuous, and how this helps you evaluate a limit. *e.g.*, evaluate:

1. $\lim_{(x,y)\to(1,1)} \frac{x^6 y}{2x^2 - y}$.
2. $\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{4 + x^2 + \cos y}$,
3. $\lim_{(x,y)\to(0,0)} \frac{x - y}{x + y}$,
4. $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}$,
5. $\lim_{(x,y)\to(0,0)} \frac{6x^3 y}{2x^4 + y^4}$.
6. $\lim_{(x,y)\to(0,0)} \frac{xy + x^2 y^2}{xy^2 + x^2 y}$.

Answer:

1. By continuity, this is 1.
2. By continuity, this is 0.
3. Does not exist.
4. Does not exist.
5. Does not exist.
6. This one has some domain problems. First try $x = y > 0$, with $x \to 0$. Then try $x = y < 0$ with $x \to 0$. You should get $\infty, -\infty$ respectively. Thus the limit does not exist. This is a considerably harder problem than the rest of them.

5. (§14.3) Given a function $f(x, y)$, find the partials $f_x, f_y, f_{xx}, f_{xy}, f_{yy}$, etc. When do we have $f_{xy} = f_{yx}$?

1. $f(x, y) = 3x + 2y + 4$
2. $f(x, y) = 5x^2 + 2xy - 3y^2 + x - y$
3. $f(x, y) = e^{xy}$
4. $f(x, y) = x^2 \sin xy^2$

Answer: Clairaut's theorem states that if the second partials f_{xy} and f_{yx} exist and are continuous, then they are equal.

1. $f_x = 3$, $f_y = 2$, $f_{xx} = 0$, $f_{yy} = 0$, $f_{xy} = f_{yx} = 0$.

2. $f_x = 10x + 2y + 1,\ f_y = 2x - 6y - 1,\ f_{xx} = 10,\ f_{yy} = -6,\ f_{xy} = f_{yx} = 2.$
3. $f_x = ye^{xy},\ f_y = xe^{xy},\ f_{xx} = y^2 e^{xy},\ f_{yy} = x^2 e^{xy},\ f_{xy} = f_{yx} = yxe^{xy} + e^{xy}.$
4. $f_x = 2x\sin xy^2 + x^2y^2\cos xy^2,\ f_y = 2x^3 y\cos xy^2,\ f_{xx} = 2\sin xy^2 + 4xy^2\cos xy^2 - x^2y^4\sin xy^2,\ f_{yy} = 2x^3\cos xy^2 - 4x^4y^2\sin xy^2,\ f_{xy} = f_{yx} = 6x^2 y\cos xy^2 - 2x^3y^3\sin xy^2.$

6. (§14.4) Given a function $f(x,y)$, and point (x_0,y_0), find the equation of the tangent plane to f at this point.

1. $f(x,y) = x^2 + 2xy + y^2, (x_0,y_0) = (1,1)$.
2. $f(x,y) = \cos\left(xy^2\right), (x_0,y_0) = (\pi,1)$.
3. $f(x,y) = e^{xy}, (x_0,y_0) = (2,3)$.
4. $f(x,y) = \sqrt{x^2+y^2}, (x_0,y_0) = (-3,4)$.

Answer: 1. $z - 4 = 4(x-1) + 4(y-1).$

2. $z = -1.$
3. $z - e^6 = 3e^6(x-2) + 2e^6(y-3).$
4. $z - 5 = (-3/5)(x+3) + (4/5)(y-4).$

7. (§14.4) Given a function $f(x,y)$, and point (x_0,y_0), find the equation of the linearization of f, call it $L(x,y)$ at the given point. Use the linearization to approximate $f(x,y)$ for some point $(x,y) \approx (x_0,y_0)$.

1. $f(x,y) = (x+y)^3, (x_0,y_0) = (1,1)$, approximate $f(0.9, 1.1)$.
2. $f(x,y) = \sqrt{xy}, (x_0,y_0) = (2,8)$, approximate $f(2.1, 8.1)$.
3. Approximate $\sin(\pi + 0.1)\cos(0.05)$.

Answer: The linearization has the form

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$$

1. $L(x,y) = 8 + 12(x-1) + 12(y-1).$ $L(0.9, 1.1) = 8.$
2. $L(x,y) = 4 + (x-2) + \frac{1}{4}(y-8) = x + \frac{1}{4}y.$ $L(2.1, 8.1) = 4.125.$
3. We are letting $f(x,y) = \sin(x)\cos(y)$. We can easily find the Linearization expanded around $(\pi, 0)$: $L(x,y) = f(\pi,0) + \nabla f(\pi,0)\cdot\langle x-\pi, y\rangle$. This gives $L(x,y) = 0 + \langle -1, 0\rangle \cdot \langle x - \pi, y\rangle = \pi - x$. Then we have $f(\pi + 0.1, 0.05) \approx L(\pi + 0.1, 0.05) = \pi - (\pi + 0.1) = -0.1$. The real answer is $-0.0997...$

8. (§14.5) Be prepared to use the chain rule to find derivatives and partial derivatives. *e.g.,*:

1. $f(x,y) = (x+y)^4, x = \sqrt{t}, y = t^2$, find $\frac{\mathrm{df}}{\mathrm{dt}}$.
2. $f(x,y) = x^2 + \cos(xy), x = s + t, y = s^2$, find $\frac{\partial f}{\partial s}$, and $\frac{\partial f}{\partial t}$.
3. $f(x,y,z) = \sqrt{x^2+y^2+z^2}, x = s\cos t, y = s\sin t, z = s^2$, find $\frac{\partial f}{\partial s}$, and $\frac{\partial f}{\partial t}$.
4. $f(x,y,z) = \left(x^a + y^b + z^c\right)^n, x = r\sin s\cos t, y = r\sin s\sin s, z = r\cos s$. Find $\frac{\partial f}{\partial r}$, $\frac{\partial f}{\partial s}$, and $\frac{\partial f}{\partial t}$.

Answer: 1. $4\left(\sqrt{t} + t^2\right)\left[\frac{1}{2\sqrt{t}} + 2t\right].$

2. $\frac{\partial f}{\partial s} = \nabla f \cdot \langle 1, 2s\rangle, \frac{\partial f}{\partial t} = \nabla f\cdot\langle 1, 0\rangle.$
3. $\frac{\partial f}{\partial s} = \frac{1}{\sqrt{s^2+s^4}}\left(s + 2s^3\right), \frac{\partial f}{\partial t} = 0.$

9. (§14.6) Given a function $f(x,y)$, or $f(x,y,z)$, find the gradient ∇f.

1. $f(x,y) = x^2y + y$.
2. $f(x,y) = \cos(xy) + x^2 + y^2$.
3. $f(x,y) = \sqrt{x^2+y^2}$.
4. $f(x,y) = \left(x^a + y^b\right)^n$.
5. $f(x,y,z) = \sin(z)\, e^{xy}$.
6. $f(x,y,z) = \sqrt{x^2+y^2-z^2}$.
7. $f(x,y,z) = \sin(xy) + \cos(yz)$.
8. $f(x,y) = \frac{xy(x^2-y^2)}{x^2+y^2}$.

Answer: 1. $\nabla f = \left\langle 2xy, x^2+1 \right\rangle$.

2. $\nabla f = \langle -\sin(xy)\, y + 2x, -\sin(xy)\, x + 2y \rangle$.
3. $\nabla f = \left\langle x/\sqrt{x^2+y^2}, y/\sqrt{x^2+y^2} \right\rangle = (1/f)\langle x, y\rangle$.
4. $\nabla f = \left\langle nax^{a-1}\left(x^a+y^b\right)^{n-1}, nby^{b-1}\left(x^a+y^b\right)^{n-1} \right\rangle$.
5. $\nabla f = \langle \sin(z)\, ye^{xy}, \sin(z)\, xe^{xy}, \cos(z)\, e^{xy} \rangle = f(x,y,z)\langle y, x, \cot z\rangle$.
6. $\nabla f = \left\langle x/\sqrt{x^2+y^2-z^2}, y/\sqrt{x^2+y^2-z^2}, -z/\sqrt{x^2+y^2-z^2} \right\rangle = (1/f(x,y,z))\langle x, y, -z\rangle$.
7. $\nabla f = \langle y\cos(xy), x\cos(xy) - z\sin(yz), -y\sin(yz) \rangle$

10. (§14.6) Given a function $f(x,y)$, and a unit vector $\boldsymbol{u}$, find the directional derivative of f in the direction of $\boldsymbol{u}$, $\mathrm{D}_{\boldsymbol{u}}f(x,y)$. *e.g.*,:

1. $f(x,y) = x^2 + xy + y^2, \boldsymbol{u} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$.
2. $f(x,y) = \cos\frac{x}{y} + \cos\frac{y}{x}, \boldsymbol{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$.
3. $f(x,y) = \arctan xy, \boldsymbol{u} = \langle 1, 0\rangle$, at the point $(x,y) = (1,1)$.
4. $f(x,y) = \sqrt{x^2+y^2}, \boldsymbol{u} = \langle \cos\theta, \sin\theta\rangle$, for some θ.

Answer: Remember that $\mathrm{D}_{\boldsymbol{u}}f(x,y) = \nabla f(x,y)\cdot\boldsymbol{u}$. This will be one of the formulæ provided on the top of your exam sheet.

1. $2x + \frac{11}{5}y$.
2. $\frac{1}{\sqrt{2}}\left(-\frac{1}{y}\sin\frac{x}{y} + \frac{y}{x^2}\sin\frac{y}{x} - \frac{1}{x}\sin\frac{y}{x} + \frac{x}{y^2}\sin\frac{x}{y}\right)$.
3. $\frac{1}{2}$.
4. $\nabla f(x,y)\cdot\langle\cos\theta, \sin\theta\rangle = (1/f(x,y))\langle x, y\rangle\cdot\langle\cos\theta,\sin\theta\rangle = (x\cos\theta + y\sin\theta)/\sqrt{x^2+y^2}$.

11. (§14.6) Understand the connection between ∇f and the directional derivative, the direction of maximal increase of f, and the maximal rate of instantaneous increase of f. *e.g.*,:

1. For $f(x,y) = \sqrt{x^2+y^2}$ at $(3,4)$, what is the direction and rate of maximal increase of f?
2. Let $f(x,y) = \sin\left(x^2y\right) + y^2$ denote the elevation of a mountain at point (x,y). A hiker is at $(0,\sqrt{\pi})$, and wishes to identify the direction of steepest ascent up the mountain. Which direction is that?
3. Captain Queeg's submarine is in trouble off the coast of Madagascar: radioactive

lava is flowing out of a vent in the sea floor. Under a local coordinate system, the concentration of radioactive stuff is given as

$$f(x, y, z) = \frac{1}{1 + (x-2)^2 + (y+1)^2 + (z-3)^2}.$$

Captain Queeg's submarine is located at $(1, 1, 2)$ under this coordinate system, and the crew is quickly being sickened from the effects of radiation. In which direction should the Captain steer the ship to experience the quickest (i.e. maximal) decrease in concentration of radioactivity?

Answer: The direction of maximal increase is a unit vector in the direction of ∇f. The maximal rate of instantaneous increase is $|\nabla f|$.

1. $\langle 3/5, 4/5 \rangle$, and the rate is 1.
2. Direction of steepest ascent is $\nabla f = \left\langle 2xy\cos\left(x^2y\right), x^2\cos\left(x^2y\right) + 2y \right\rangle$. At the point $(0, \sqrt{\pi})$ this is $\nabla f\left(0, \sqrt{\pi}\right) = \langle 0, 2\sqrt{\pi} \rangle$. Thus the direction of steepest ascent is the unit vector in this direction, $\langle 0, 1 \rangle$.
3. the direction of maximal decrease in f is the unit vector in direction $-\nabla f$. Thus we find

$$\nabla f\left(x, y, z\right) = -\left(f\left(x, y, z\right)\right)^2 \langle 2(x-2), 2(y+1), 2(z-3) \rangle.$$

This gives $\nabla f\left(1, 1, 2\right) = -\frac{1}{(1+(-1)^2+2^2+(-1)^2)^2} \langle 2(-1), 2(2), 2(-1) \rangle = (2/49) \langle -1, 2, -1 \rangle$. The unit vector in the same direction is $\left(1/\sqrt{6}\right) \langle -1, 2, -1 \rangle$.

12. (§14.6) Understand the connection between ∇f and a level set of f. *e.g.*,:

1. Find the equation of the plane tangent to the surface $x^2 + y^2 + z^2 = 16$ at the point $\left(2, 2\sqrt{2}, 2\right)$.
2. Find the equation of the plane tangent to the surface $x^2 + y^2 - z = 9$ at the point $(3, 2, 4)$.
3. Find the equation of the plane tangent to the surface $a^2x^2 + b^2y^2 + c^2z^2 = 3r^2$ at the point $(r/a, r/b, r/c)$.

Answer: View the surface as a level set $f(x, y, z) = k$, then find gradient $\nabla f(x_0, y_0, z_0)$. This vector is the plane's normal. Plug in the point to get the equation of the plane.

1. $4x + 4\sqrt{2}y + 4z - 32 = 0$
2. $6x + 4y - z - 22 = 0$
3. $2arx + 2bry + 2crz - 6r^2 = 0$, or, more simply, $ax + by + cz = 3r$.

13. (§14.7) Given a function $f(x, y)$, be prepared to find all its critical points. Be able to apply the second derivative test to see if you have a max, min, or neither. Be able to find extremal values of $f(x, y)$ on a closed, bounded domain D. *e.g.*,:

1. $f(x, y) = x^4 + y^4 - 4xy$.
2. $f(x, y) = (1 + xy)(x + y)$.
3. $f(x, y) = e^x \cos y$.
4. $f(x, y) = x + 3y - 4$ on the triangle with corners $(0, 0)$,$(1, 0)$,$(0, 1)$.
5. $f(x, y) = 2x^2 + 3y^2 - 4x - 5$ on the disc $x^2 + y^2 \leq 16$.

Answer: A critical point is a point where ∇f either does not exist or is equal to $\mathbf{0}$. In the latter case we can use the second derivative test, if the second derivatives exist and are continuous, then we can evaluate $D = f_{xx}f_{yy} - f_{xy}^2$. If this quantity is positive, then we have a max or min. If it is negative, we have a saddle point. If the quantity is positive, we go on to check the sign of f_{xx}. If it is positive, we have a min, otherwise a max.

1. Local minima at $(-1,-1)$ and $(1,1)$. Saddle point at $(0,0)$.
2. $f(x,y) = x + y + x^2y + xy^2$, so $\nabla f = \langle 1 + 2xy + y^2, 1 + 2xy + x^2 \rangle$. The critical points occur where $\nabla f = \mathbf{0}$. This gives two equations in two unknowns:

   $$1 + 2xy + y^2 = 0 \quad \text{and} \quad 1 + 2xy + x^2 = 0.$$

   Set both right hand sides equal to each other: $1 + 2xy + y^2 = 1 + 2xy + x^2$, and subtract to get $y^2 = x^2$, which is to say $y = \pm x$. Plug this back into the original equation:

   $$1 + 2x(\pm x) + x^2 = 0$$

   This gives two quadratic equations in x depending on whether $y = +x$ or $y = -x$. That is, we get

   $$1 + 3x^2 = 0 \quad \text{and} \quad 1 - x^2 = 0$$

   The former equation has only imaginary roots, so this leaves the latter, which has roots $x = \pm 1$. This gives us critical points $(1,-1)$ and $(-1,1)$.
   The discriminant is $D = 4xy - (2x + 2y)^2 = 4xy - 4x^2 - 8xy - 4y^2$. This is negative for both critical points, thus they are saddle points both.
3. This has no critical points. Thus no extrema on the unbounded set $\mathbb{R}^2$.
4. This has no critical points. Min of -4 at $(0,0)$, and max of -1 at $(0,1)$.
5. I did this one in class.

14. (§14.8) Given a function $f(x,y)$, find the extremal values of f subject to the constraint $g(x,y) = k$ by the method of Lagrange Multipliers.

1. $f(x,y) = x + xy + y$, subject to $x^2 + y^2 = 9$.
2. $f(x,y) = \frac{1}{xy}$, subject to $x + y = 4$.
3. $f(x,y,z) = 8x - 4z$, subject to $x^2 + 10y^2 + z^2 = 5$.
4. $f(x,y,z) = xyz$, subject to $2xz + 2yz + xy = 12$.

Answer: Lagrange's method: Find x, y, λ such that $\nabla f(x,y) = \lambda g(x,y)$ and $g(x,y) = k$. After finding all such (x,y), tabulate your results and compare the values of f.

1. At $\left(-\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right)$, takes min value $\frac{9}{2} - \frac{6}{\sqrt{2}}$. At $\left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$, takes max value $\frac{9}{2} + \frac{6}{\sqrt{2}}$.
2. This is a poor Lagrange question, but we expect a local min of $\frac{1}{4}$ at $(2,2)$.
3. At $(-2,1)$ takes min value -20. At $(2,-1)$ takes max value 20
4. I did something like this in class.

15. (§15.1) Given a function $f(x,y)$ and a rectangle $R = [a,b] \times [c,d]$, define the integral

$$\iint_R f(x,y)\,\mathrm{d}A$$

Answer: The integral is the limit of the Riemann sums

$$\lim_{m\to\infty,\, n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f\left(x_{i,j}^*, y_{i,j}^*\right) \Delta x \Delta y,$$

where $\Delta x = (b-a)/m$, and $\Delta y = (d-c)/n$, and

$$a+(i-1)\Delta x = x_{i-1} \le x_{i,j}^* \le x_i = a+i\Delta x \quad \text{and} \quad c+(j-1)\Delta y = y_{j-1} \le y_{i,j}^* \le y_i = c+j\Delta y.$$

16. (§15.1) Estimate the integral $\iint\limits_R f(x,y)\,\mathrm{d}A$ by a Riemann Sum.

1. Do this for $f(x,y) = x^2 - xy^2$ for $R = [1,2] \times [1,4]$ with $m = 2, n = 3$. Take $\left(x_{i,j}^*, y_{i,j}^*\right)$ to be the lower left corner of each rectangle.
2. Do this for $f(x,y) = x^2 - 2xy + y^2$ for $R = [-1,1] \times [-1,1]$ with $m = 2, n = 2$. Take $\left(x_{i,j}^*, y_{i,j}^*\right)$ to be the midpoint of each rectangle.

Answer: 1. With $\Delta x = (2-1)/2 = 1/2$, $\Delta y = (4-1)/3 = 1$, we have $x_0 = 1, x_1 = 3/2, x_2 = 2$, and $y_0 = 1, y_1 = 2, y_2 = 3, y_3 = 4$. Then our Riemann Sum is

$$\begin{aligned} &\sum_{i=1}^{2} \sum_{j=1}^{3} f\left(x_{i,j}^*, y_{i,j}^*\right)(1/2)(1) \\ &= \frac{1}{2}\left[f(1,1) + f(1,2) + f(1,3) + f(3/2,1) + f(3/2,2) + f(3/2,3)\right] \\ &= \frac{1}{2}\left[(1-1) + (1-4) + (1-9) + (9/4-3/2) + (9/4-6) + (9/4-27/2)\right] \\ &= \frac{1}{2}\left[0-3-8+27/4-6-30/2\right] = \frac{1}{2}\left[-11+6+3/4-6-15\right] = -101/8 \end{aligned}$$

2. The answer is 2.

17. (misc.) 1. On your first day working in the cardboard box factory, you are given the problem of minimizing the cost of your company's next generation product: the cardboard box. This cardboard box of the future consists of a bottom, four sides and a top, all of which are rectangular. The bottom is made of a special cardboard, called KardBord MIII™, which costs 5 cents per square inch. The sides and the top are made out of second generation recycled cardboard, called ReSyKleBoRd 2™, which costs 3 cents per square inch. The total volume of the box is to be 972 cubic inches. Due to restrictions beyond your control (*i.e.*, dictated by UPS and FedEx), the width of the box may not exceed 7 inches. Find the minimal cost size box. Your next assignment: find a better job than working in a cardboard factory.

2. Let a, b, c be positive integers. Find positive real numbers x, y, z such that $x+y+z = 117$ and such that $x^a y^b z^c$ is maximized.
3. Consider the following party trick: a spherical balloon of radius r is placed in the oven at 250 degrees for 10 minutes. The resultant increase in temperature causes the radius to increase to $r + \Delta r$. Estimate the change in the volume of the balloon. What is the answer if $r = 10$ cm and $\Delta r = 1$ cm?

4. You are the new chief operating officer at `www.ohmcalc.com`, the newest way to calculate resistance over the internet. Registered (*i.e.*, paying) users have the capability of entering, via the web, the resistance of two resistors; your site computes the total resistance of these resistors when placed in parallel, using the following formula:

$$\frac{1}{R_{total}} = \frac{1}{R_1} + \frac{1}{R_2}$$

The measured resistance of resistors, alas, comes with some error, thus you need to estimate the error in your calcluated total resistance. Estimate the error in your calculation when $R_1 = 500\Omega \pm 25\Omega$, and $R_2 = 3000\Omega \pm 100\Omega$. (If this doesn't work out for you, perhaps you might consider a career in cardboard box manufacturing...)

Answer: 1. Let x, y be the width and the length, and let $972/xy$ be the height. We wish to minimize

$$f(x, y) = 5xy + 3\left[2(x972/xy) + 2(y972/xy) + xy\right] = 8xy + 5832\left[\frac{1}{y} + \frac{1}{x}\right]$$

We also have the restrictions $x \in (0, 7]$, and $y \in (0, \infty)$
We find the gradient

$$\nabla f(x, y) = \left\langle 8y - 5832/x^2, 8x - 5832/y^2 \right\rangle$$

and set equal to $\mathbf{0}$. This gives two equations, which we set equal to each other:

$$\begin{aligned} 8y - 5832/x^2 &= 0 = 8x - 5832/y^2, \quad \text{so} \\ 8x^2y^3 - 5832y^2 &= 8x^3y^2 - 5832x^2, \quad \text{or} \\ 0 &= 8x^2y^2(x - y) - 5832(x^2 - y^2) \\ 0 &= 8x^2y^2(x - y) - 5832(x + y)(x - y) = (x - y)\left[8x^2y^2 - 5832(x + y)\right] \end{aligned}$$

This gives two possibilities: $x - y = 0$ or $x^2y^2 = 729(x + y)$. The latter condition cannot be satisfied, as we have

$$\begin{aligned} 0 &= 8y - 5832/x^2, \quad \text{so} \\ y &= 729/x^2, \\ y^2x^2 &= 729y, \end{aligned}$$

Subtracting this from $x^2y^2 = 729x + 729y$ gives $729x = 0$, or $x = 0$. Similar argument starting from $0 = 8x - 5832/y^2$ gives $y = 0$, and so the latter condition implies $x = y$ anyway.
Thus we can plug in $x = y$ into the equation

$$0 = 8y - 5832/x^2 = 8x - 5832/x^2,$$

which is to say $x^3 = 729$, or $x = 9$. This gives $y = 9$ immediately as well. This is a local (and global!) min, by checking the second derivative test. It is also a bummer because the requirement that $x \leq 7$. Thus we have to check the boundaries of our region.

Our region is $R = (0, 7] \times (0, \infty)$, which is neither closed nor bounded. We didn't really look at this in class so much, but we can consider it here. Since there is no critical point in R, we only have to check its boundaries. It should be clear that as x or y goes to 0 that $f(x, y) \to \infty$. Similarly as $y \to \infty$, f explodes. Since we are minimizing cost we do not need to consider these. Thus we look at the boundary $x = 7$. This gives $g(y) = f(7, y) = 56y + 5832(1/7 + 1/y)$, which we have to minimize. Use your 20A knowledge: $g'(y) = 56 - 5832/y^2$. Set this to zero to get the quadratic equation $0 = 7 - 729/y^2$, or $7y^2 = 729$, which has solution $y = \sqrt{729/7}$. By the second derivative test, this is a local min, so we have $\left(7, \sqrt{729/7}\right)$ as the local min. This question was a little harder than we did in class, mostly because of the hand-wavey parts where the boundary is not closed and bounded. If you like, you could try the question where I explicitly mention in the problem formulation that we must take $(x, y) \in [1, 7] \times [1, 1000]$. Then you could do this exactly how we did it in class, but with more work than we did here because you would have to consider all four boundary edges separately, instead of throwing three of them away as we did here with the argument that $f \to \infty$.

2. Let $z = 117 - x - y$ and let $f(x, y) = x^a y^b (117 - x - y)^c$. Find the critical points:

$$\begin{aligned} \nabla f &= (\text{work omitted}) \cdots \\ &= x^{a-1} y^{b-1} (117 - x - y)^{c-1} \langle ay(117 - x - y) - cxy, bx(117 - x - y) - cxy \rangle \end{aligned}$$

Setting this to $\mathbf{0}$, and assuming $x, y, 117 - x - y$ are not zero, gives two equations:

$$ay(117 - x - y) = cxy, \quad \text{and} \quad bx(117 - x - y) = cxy$$

By setting these two equal to each other and cancelling, we get $ay = bx$, or $y = (b/a)x$. We also get, from the first equation, cancelling y, $cx = a(117 - x - y) = 117a - ax - bx$, and thus $x = 117a/(a + b + c)$, and $y = 117b/(a + b + c)$, which gives $z = 117 - x - y = 117c/(a + b + c)$.

3. (OOOPPS! I just realized this question is irrelevant for 20C. Because there is only one variable, it is not a differential of two variables question. Bummer.)

4. Use differentials. First rewrite

$$R_{total} = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)^{-1} = \frac{R_1 R_2}{R_2 + R_1}$$

Now get the differential

$$\mathrm{d}R_{total} = \left(\frac{R_2}{R_1 + R_2}\right)^2 \mathrm{d}R_1 + \left(\frac{R_1}{R_1 + R_2}\right)^2 \mathrm{d}R_2$$

Now approximate

$$\begin{aligned} \Delta R_{total} \approx \mathrm{d}R_{total} &\approx \left(\frac{R_2}{R_1 + R_2}\right)^2 \Delta R_1 + \left(\frac{R_1}{R_1 + R_2}\right)^2 \Delta R_2 \\ &= \left(\frac{3000}{3500}\right)^2 25 + \left(\frac{500}{3500}\right)^2 100 \\ &= \frac{1000}{49} \Omega \end{aligned}$$