2005S M20C Final Exam Preparation

The final exam is Friday June 10, from 8:00am-11:00am. The exam *will be held in the Mandeville Auditorium,* in the Mandeville building. This is *not* our regular meeting place. You are warned. *You must bring a blue book to the exam.* Blue books are available from the bookstore. You also need to bring your student ID card or other form of ID (driver's license, passport, etc.) The exam is comprehensive: it covers all material from the quarter. There will be slightly more emphasis on material covered since the second midterm, that is §15.2–15.5, 15.7, 12.7, and whatever of §15.8 we get to in class on the last day.

You will have three hours to complete the final exam. It is, however, only about 50% longer than either midterm. So you should have plenty of time to complete the exam. The final exam is worth approximately 60% more than either of the midterms, thus it is based on 160 points. Thus one point on the final is about equal to one point on a midterm. A question which was worth about 15 points on midterm 1 would be worth the same on the final.

You may not use a calculator or notes during the exam.

The following formulæ will be provided on your exam:

$$\boldsymbol{a}\cdot\boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3 \qquad \boldsymbol{a}\times\boldsymbol{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle$$

$$|\boldsymbol{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2} \qquad L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$$s = \int \left|\boldsymbol{R}'(t)\right| \,\mathrm{d}t \qquad \mathrm{D}_{\boldsymbol{u}}f = \nabla f\cdot\boldsymbol{u} \qquad \nabla f = \langle f_x, f_y\rangle \qquad D = f_{xx}f_{yy} - f_{yx}f_{xy}$$

$$\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta \qquad \mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z = \rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$$

Everything else must be committed to memory.

This exam preparation sheet is comprehensive. It includes the two midterm prep sheets as well as a prep sheet for material since the last midterm.

You should be prepared to answer at least the following questions:

1. What's the difference between a scalar and a vector? Describe the input and output types of common operations.

2. (§12.1) Let $P = (2,3,1)\,, Q = (4,-2,5)$.

1. What is the "position vector" of P?
2. Find the distance from P to Q.
3. Find the vector from P to Q, $\boldsymbol{PQ}$.
4. Find the distance from P to the origin.
5. Find the distance from P to the xy plane.
6. Find the equation of the sphere centered at P with radius 5.
7. Find the equation of the sphere centered at Q such that P is on the sphere.

3. (§12.2–12.5) Let $\boldsymbol{a} = \langle 1,0,1\rangle\,, \boldsymbol{b} = \langle 1,3,0\rangle\,.$

1. Find $|\boldsymbol{a}|$.
2. Find a unit vector in the same direction as $\boldsymbol{a}$.

3. Find $\boldsymbol{a} + \boldsymbol{b}$.
4. Find $3\boldsymbol{a} - 2\boldsymbol{b}$.
5. Find $\boldsymbol{a} \cdot \boldsymbol{b}$.
6. Find $\boldsymbol{a} \times \boldsymbol{b}$.
7. Find a vector perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$.
8. Find the angle subtended by the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.
9. Find the component of $\boldsymbol{a}$ along $\boldsymbol{b}$.
10. Find the projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$.
11. Give a parametrization of the line through the point $(3, 4, 1)$ parallel to $\boldsymbol{b}$.
12. Give an equation of the plane through the origin that is parallel to both $\boldsymbol{a}$ and $\boldsymbol{b}$.
13. Give an equation of the plane containing the point $(1, 2, -1)$ that has $\boldsymbol{a}$ as a normal.
14. Find the angle subtended by $\boldsymbol{a}$ and $\boldsymbol{a} \times \boldsymbol{b}$.
15. Is $\boldsymbol{a}$ parallel to $\boldsymbol{b}$? Are they perpendicular?
16. Find a vector parallel to $\boldsymbol{a}$.

4. (§12.2–12.5) Let $\boldsymbol{u} = \langle 1, 2, -3 \rangle$, $\boldsymbol{v} = \langle -2, 4, 1 \rangle$, $\boldsymbol{w} = \langle -3, 0, 1 \rangle$.

1. Find $5\boldsymbol{u}$.
2. Find $\boldsymbol{u} \cdot \boldsymbol{u}$.
3. Find $|\boldsymbol{u}|$.
4. Find a unit length vector in the same direction as $\boldsymbol{u}$.
5. Find $|\alpha \boldsymbol{u}|$, where α is a real number.
6. Find $\boldsymbol{u} \cdot \boldsymbol{v}$.
7. Find the angle subtended by $\boldsymbol{u}$ and $\boldsymbol{v}$.
8. Find $\boldsymbol{v} \times \boldsymbol{w}$.
9. Find a vector perpendicular to both $\boldsymbol{v}$ and $\boldsymbol{w}$.
10. Find a unit length vector perpendicular to both $\boldsymbol{v}$ and $\boldsymbol{w}$.
11. Find $\boldsymbol{u} \times \boldsymbol{v}$.
12. Find $\boldsymbol{w} \cdot (\boldsymbol{u} \times \boldsymbol{v})$.
13. Find $\boldsymbol{u} \times \boldsymbol{u}$.

5. (§12.5) Let $\boldsymbol{R_1}(t) = \langle 0, 3, 1 \rangle + t \langle 2, 1, -1 \rangle$ parametrize the line l_1. Let $\boldsymbol{R_2}(t) = \langle 1, 2, 1 \rangle + t \langle 2, 3, 0 \rangle$ parametrize the line l_2. Let $3x - 2y + 5z = 1$ describe a plane P.

1. Find a vector parallel to l_1. Find a vector parallel to l_2.
2. Find a vector normal to P.
3. Do the lines l_1, l_2 intersect? If so, find their point of intersection. If not, are they parallel, or are they skew?
4. Find the intersection of l_1 and P.
5. Find the intersection of l_2 and P. (*Hint:* it's a trick)
6. Find the angle that l_1 subtends with the plane P.
7. Find the distance from l_1 to the point $(2, 2, 1)$.

8. Find the distance from l_2 to P.

6. (§12.5) Let $3x - 2y + 5z = 1$ describe a plane P. Let $x + y + 3z = 4$ describe a plane Q.

1. Find a vector normal to P.
2. Find a vector normal to Q.
3. Are the planes P, Q parallel? If not, must they intersect?
4. Find the angle subtended by the two planes

7. (§13.1) Let $\boldsymbol{u} = \langle -2, 1, 4 \rangle$, and let P be the point $(0, 4, 1)$.

1. Find a vector valued function $\boldsymbol{l}(t)$ which traces out the line through P and parallel to $\boldsymbol{u}$.
2. Find the equation of the plane containing P with normal $\boldsymbol{u}$.
3. Find the parametrization, $\boldsymbol{m}(t)$, of the line line through P and perpendicular to $\boldsymbol{u}$. (*Hint:* There are many answers. One way to do this is pick some arbitrary vector $\boldsymbol{v}$ not parallel to $\boldsymbol{u}$, then find a vector perpendicular to both $\boldsymbol{u}$ and $\boldsymbol{v}$.)

8. (§13.2) Find the limits:

1. $\lim_{t \to 0} \boldsymbol{R}(t)$, where $\boldsymbol{R}(t) = \left\langle t + 1, 5, t^2 \right\rangle$.
2. $\lim_{t \to 0} \left\langle \frac{\sin t}{t}, e^{-t}, t \right\rangle$.
3. $\lim_{t \to 0} \left\langle \frac{t^2 + t^4}{4t^2 - t^3}, 5t + 4, \ln(1 + t) \right\rangle$.
4. $\lim_{t \to 1} \left\langle \frac{1 - t}{1 - t^2}, \frac{\ln t}{1 - t}, 5t \right\rangle$.

9. (§13.2) Given $\boldsymbol{R}(t)$, find its antiderivative $\int \boldsymbol{R}(t)\,\mathrm{d}t$.

1. $\boldsymbol{R}(t) = \left\langle 2t - 1, t^2, \cos t \right\rangle$.
2. $\boldsymbol{R}(t) = \left\langle \frac{1}{1 + t^2}, \frac{1}{t}, \sqrt{2t + 1} \right\rangle$.

10. (§13.2) Given vector valued functions $\boldsymbol{R}(t)$ and $\boldsymbol{S}(t)$, which describe the position of two particles, find whether the particles collide. Find if the paths of the two particles intersect.

1. $\boldsymbol{R}(t) = \left\langle 2t, t^2, \cos t \right\rangle$, $\boldsymbol{S}(t) = \left\langle \sin t, -e^t, 4t \right\rangle$.
2. $\boldsymbol{R}(t) = \langle 1 + 3t, 2 - 5t, -2 + t \rangle$, $\boldsymbol{S}(t) = \langle 5 - t, 1 - 4t, -3 + 2t \rangle$.
3. $\boldsymbol{R}(t) = \langle 1 + t, 2 + t, 3 + t \rangle$, $\boldsymbol{S}(t) = \left\langle t - 1, t^2, t + 1 \right\rangle$.

11. (§13.2) Find $\boldsymbol{R}'(t)$ for

1. $\boldsymbol{R}(t) = \langle t, 3 - 2t, 4 + 6t \rangle$.
2. $\boldsymbol{R}(t) = \left\langle t + 1, 5, t^2 \right\rangle$.
3. $\boldsymbol{R}(t) = \langle 4, 7, 1 \rangle \times \left\langle t + 1, 5, t^2 \right\rangle$.
4. $\boldsymbol{R}(t) = \langle \arctan t, \sin t, \cos t \rangle$.
5. $\boldsymbol{R}(t) = \left\langle \arctan\left(e^t\right), \sin\left(e^t\right), \cos\left(e^t\right) \right\rangle$.
6. $\boldsymbol{R}(t) = \langle \cos t, \sin t, 3t \rangle$.
7. $\boldsymbol{R}(t) = \left\langle (t + 1)\sin t, 5\sin t, t^2 \sin t \right\rangle$.

12. (§13.3) Given vector valued function $\boldsymbol{R}(t)$, find the arc length of the curve traced by $\boldsymbol{R}$ for t between t_0 and t_1, *e.g.*,

1. $\boldsymbol{R}(t) = \langle 1+3t, 3, 1-4t \rangle$, $t_0 = 0$, $t_1 = 3$.
2. $\boldsymbol{R}(t) = \langle 2\cos t, 2\sin t \rangle, 0 \le t \le \pi$.
3. $\boldsymbol{R}(t) = \langle \cos t, \sin t, 3t \rangle$, $t_0 = 0$, $t_1 = 4\pi$.
4. $\boldsymbol{R}(t) = \left\langle t/\sqrt{2}, t/\sqrt{2}, \ln(\sec t) \right\rangle$, $t_0 = 0$, $t_1 = \pi/3$.
5. $\boldsymbol{R}(t) = \left\langle t, t^3/3 + 1/(4t) \right\rangle$, $t_0 = 1$, $t_1 = 2$.
6. $\boldsymbol{R}(t) = \left\langle \sqrt{2}t, e^t, e^{-t} \right\rangle, 0 \le t \le 1$.
7. $\boldsymbol{R}(t) = \left\langle t^2, t, 3 \right\rangle, 0 \le t \le 1$.

13. (§13.4) Given the position of a particle, $\boldsymbol{R}(t)$, find its velocity, speed, and acceleration. *e.g.,*:
1. $\boldsymbol{R}(t) = \langle 2\cos t, 2\sin t \rangle$.
2. $\boldsymbol{R}(t) = \left\langle \sqrt{2}t, e^t, e^{-t} \right\rangle$.
3. $\boldsymbol{R}(t) = \left\langle t^2, t, 3 \right\rangle$.

14. (§13.4) Given the acceleration of a particle and its initial position and velocity, find the position of the particle. *e.g.,*:
1. $\boldsymbol{a}(t) = \langle 1, t, \sin t \rangle, \boldsymbol{v}(0) = \langle 0, 1, 0 \rangle, \boldsymbol{R}(0) = \langle 0, 0, 0 \rangle$.
2. $\boldsymbol{a}(t) = \langle \sin t, \cos t \rangle, \boldsymbol{v}(0) = \langle 1, 0 \rangle, \boldsymbol{R}(0) = \langle 0, 1 \rangle$.

15. (§14.1) You should have some basic understanding of functions of two or more variables. *e.g.,*: given $f(x, y)$, be able to evaluate $f(1, 2)$; find the domain of f; sketch a graph of $f(x, y)$, or the contours (level sets) of $f(x, y)$.
1. What is the domain of $f(x, y) = \frac{\sqrt{x-y}}{y}$?
2. Graph the level sets of $f(x, y) = \frac{1}{4}x^2 + y^2$.
3. Graph the function $f(x, y) = \sqrt{x^2 + y^2}$.
4. Graph the function and level sets of $f(x, y) = 3x - 4y + 2$.

16. (§14.1) (this is a basic one) You should understand functions of two or more variables. *e.g.,*: given $f(x, y)$, be able to evaluate $f(1, 2)$, or $f(\pi, 100)$; find the domain of f; sketch a graph of $f(x, y)$, or the contours (level sets) of $f(x, y)$.
1. What is the domain of $f(x, y) = \frac{\sqrt{x-y}}{y}$?
2. What is the domain of $f(x, y) = \frac{xy^3}{x^2+y^2}$?
3. Graph the level sets of $f(x, y) = \frac{1}{4}x^2 + y^2$.
4. Graph the function $f(x, y) = \sqrt{x^2 + y^2}$.

17. (§14.1) Sketch the level sets of $f(x, y)$ for various values of k
1. $f(x, y) = x^2 + y^2$
2. $f(x, y) = 2x - 4y + 7$
3. $f(x, y) = \sqrt{x^2 + y^2}$

18. (§14.2) Identify if a function $f(x, y)$ is continuous at a given point (x, y). Identify if the function is continuous for all points in $\mathbb{R}^2$.
1. $f(x, y) = x^2 + 4x^3y^2$ at $(x, y) = (1, 3)$.

2. $f(x,y) = \begin{cases} \frac{x+y}{x^2+y^2} & (x,y) \neq (0,0) \\ 1 & (x,y) = (0,0) \end{cases}$ at $(x,y) = (0,0)$.

3. $f(x,y) = \frac{x^3+y^3}{\sqrt{x+y}}$ at $(x,y) = (2,1)$.

19. (§14.2) You should be able to find the limit of a function of two variables, or recognize that the limit does not exist. You need to know what functions are continuous, and how this helps you evaluate a limit. *e.g.*, evaluate:

1. $\lim_{(x,y)\to(1,1)} \frac{x^6 y}{2x^2-y}$.
2. $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{4+x^2+\cos y}$,
3. $\lim_{(x,y)\to(0,0)} \frac{x-y}{x+y}$,
4. $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$,
5. $\lim_{(x,y)\to(0,0)} \frac{6x^3 y}{2x^4+y^4}$.
6. $\lim_{(x,y)\to(0,0)} \frac{xy+x^2y^2}{xy^2+x^2y}$.

20. (§14.3) Given a function $f(x,y)$, find the partials $f_x, f_y, f_{xx}, f_{xy}, f_{yy}$, etc. When do we have $f_{xy} = f_{yx}$?

1. $f(x,y) = 3x + 2y + 4$
2. $f(x,y) = 5x^2 + 2xy - 3y^2 + x - y$
3. $f(x,y) = e^{xy}$
4. $f(x,y) = x^2 \sin xy^2$

21. (§14.4) Given a function $f(x,y)$, and point (x_0, y_0), find the equation of the tangent plane to f at this point.

1. $f(x,y) = x^2 + 2xy + y^2, (x_0, y_0) = (1,1)$.
2. $f(x,y) = \cos\left(xy^2\right), (x_0, y_0) = (\pi, 1)$.
3. $f(x,y) = e^{xy}, (x_0, y_0) = (2,3)$.
4. $f(x,y) = \sqrt{x^2+y^2}, (x_0, y_0) = (-3,4)$.

22. (§14.4) Given a function $f(x,y)$, and point (x_0, y_0), find the equation of the linearization of f, call it $L(x,y)$ at the given point. Use the linearization to approximate $f(x,y)$ for some point $(x,y) \approx (x_0, y_0)$.

1. $f(x,y) = (x+y)^3, (x_0, y_0) = (1,1)$, approximate $f(0.9, 1.1)$.
2. $f(x,y) = \sqrt{xy}, (x_0, y_0) = (2,8)$, approximate $f(2.1, 8.1)$.
3. Approximate $\sin(\pi + 0.1)\cos(0.05)$.

23. (§14.5) Be prepared to use the chain rule to find derivatives and partial derivatives. *e.g.*,:

1. $f(x,y) = (x+y)^4, x = \sqrt{t}, y = t^2$, find $\frac{\mathrm{df}}{\mathrm{dt}}$.
2. $f(x,y) = x^2 + \cos(xy), x = s+t, y = s^2$, find $\frac{\partial f}{\partial s}$, and $\frac{\partial f}{\partial t}$.
3. $f(x,y,z) = \sqrt{x^2+y^2+z^2}, x = s\cos t, y = s\sin t, z = s^2$, find $\frac{\partial f}{\partial s}$, and $\frac{\partial f}{\partial t}$.

4. $f(x,y,z) = \left(x^a + y^b + z^c\right)^n$, $x = r \sin s \cos t$, $y = r \sin s \sin s$, $z = r \cos s$. Find $\frac{\partial f}{\partial r}$, $\frac{\partial f}{\partial s}$, and $\frac{\partial f}{\partial t}$.

24. (§14.6) Given a function $f(x,y)$, or $f(x,y,z)$, find the gradient ∇f.

1. $f(x,y) = x^2 y + y$.
2. $f(x,y) = \cos(xy) + x^2 + y^2$.
3. $f(x,y) = \sqrt{x^2 + y^2}$.
4. $f(x,y) = \left(x^a + y^b\right)^n$.
5. $f(x,y,z) = \sin(z)\, e^{xy}$.
6. $f(x,y,z) = \sqrt{x^2 + y^2 - z^2}$.
7. $f(x,y,z) = \sin(xy) + \cos(yz)$.
8. $f(x,y) = \frac{xy(x^2-y^2)}{x^2+y^2}$.

25. (§14.6) Given a function $f(x,y)$, and a unit vector $\boldsymbol{u}$, find the directional derivative of f in the direction of $\boldsymbol{u}$, $\mathrm{D}_{\boldsymbol{u}} f(x,y)$. *e.g.*,:

1. $f(x,y) = x^2 + xy + y^2, \boldsymbol{u} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$.
2. $f(x,y) = \cos\frac{x}{y} + \cos\frac{y}{x}, \boldsymbol{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$.
3. $f(x,y) = \arctan xy, \boldsymbol{u} = \langle 1, 0 \rangle$, at the point $(x,y) = (1,1)$.
4. $f(x,y) = \sqrt{x^2 + y^2}, \boldsymbol{u} = \langle \cos\theta, \sin\theta \rangle$, for some θ.

26. (§14.6) Understand the connection between ∇f and the directional derivative, the direction of maximal increase of f, and the maximal rate of instantaneous increase of f. *e.g.*,:

1. For $f(x,y) = \sqrt{x^2 + y^2}$ at $(3,4)$, what is the direction and rate of maximal increase of f?
2. Let $f(x,y) = \sin\left(x^2 y\right) + y^2$ denote the elevation of a mountain at point (x,y). A hiker is at $(0, \sqrt{\pi})$, and wishes to identify the direction of steepest ascent up the mountain. Which direction is that?
3. Captain Queeg's submarine is in trouble off the coast of Madagascar: radioactive lava is flowing out of a vent in the sea floor. Under a local coordinate system, the concentration of radioactive stuff is given as

$$f(x,y,z) = \frac{1}{1 + (x-2)^2 + (y+1)^2 + (z-3)^2}.$$

Captain Queeg's submarine is located at $(1,1,2)$ under this coordinate system, and the crew is quickly being sickened from the effects of radiation. In which direction should the Captain steer the ship to experience the quickest (i.e. maximal) decrease in concentration of radioactivity?

27. (§14.6) Understand the connection between ∇f and a level set of f. *e.g.*,:

1. Find the equation of the plane tangent to the surface $x^2 + y^2 + z^2 = 16$ at the point $\left(2, 2\sqrt{2}, 2\right)$.
2. Find the equation of the plane tangent to the surface $x^2 + y^2 - z = 9$ at the point $(3,2,4)$.

3. Find the equation of the plane tangent to the surface $a^2x^2 + b^2y^2 + c^2z^2 = 3r^2$ at the point $(r/a, r/b, r/c)$.

28. (§14.7) Given a function $f(x, y)$, be prepared to find all its critical points. Be able to apply the second derivative test to see if you have a max, min, or neither. Be able to find extremal values of $f(x, y)$ on a closed, bounded domain D. *e.g.*,:

1. $f(x, y) = x^4 + y^4 - 4xy$.
2. $f(x, y) = (1 + xy)(x + y)$.
3. $f(x, y) = e^x \cos y$.
4. $f(x, y) = x + 3y - 4$ on the triangle with corners $(0, 0)$,$(1, 0)$,$(0, 1)$.
5. $f(x, y) = 2x^2 + 3y^2 - 4x - 5$ on the disc $x^2 + y^2 \leq 16$.

29. (§14.8) Given a function $f(x, y)$, find the extremal values of f subject to the constraint $g(x, y) = k$ by the method of Lagrange Multipliers.

1. $f(x, y) = x + xy + y$, subject to $x^2 + y^2 = 9$.
2. $f(x, y) = \frac{1}{xy}$, subject to $x + y = 4$.
3. $f(x, y, z) = 8x - 4z$, subject to $x^2 + 10y^2 + z^2 = 5$.
4. $f(x, y, z) = xyz$, subject to $2xz + 2yz + xy = 12$.

30. (§15.1) Given a function $f(x, y)$ and a rectangle $R = [a, b] \times [c, d]$, define the integral

$$\iint\limits_R f(x, y)\, \mathrm{d}A$$

31. (§15.1) Estimate the integral $\iint\limits_R f(x, y)\, \mathrm{d}A$ by a Riemann Sum.

1. Do this for $f(x, y) = x^2 - xy^2$ for $R = [1, 2] \times [1, 4]$ with $m = 2, n = 3$. Take $\left(x^*_{i,j}, y^*_{i,j}\right)$ to be the lower left corner of each rectangle.
2. Do this for $f(x, y) = x^2 - 2xy + y^2$ for $R = [-1, 1] \times [-1, 1]$ with $m = 2, n = 2$. Take $\left(x^*_{i,j}, y^*_{i,j}\right)$ to be the midpoint of each rectangle.

32. (misc.)

1. On your first day working in the cardboard box factory, you are given the problem of minimizing the cost of your company's next generation product: the cardboard box. This cardboard box of the future consists of a bottom, four sides and a top, all of which are rectangular. The bottom is made of a special cardboard, called KardBord MIII™, which costs 5 cents per square inch. The sides and the top are made out of second generation recycled cardboard, called ReSyKleBoRd 2™, which costs 3 cents per square inch. The total volume of the box is to be 972 cubic inches. Due to restrictions beyond your control (*i.e.*, dictated by UPS and FedEx), the width of the box may not exceed 7 inches. Find the minimal cost size box. Your next assignment: find a better job than working in a cardboard factory.
2. Let a, b, c be positive integers. Find positive real numbers x, y, z such that $x+y+z = 117$ and such that $x^a y^b z^c$ is maximized.
3. You are the new chief operating officer at `www.ohmcalc.com`, the newest way to calculate resistance over the internet. Registered (*i.e.*, paying) users have the capability

of entering, via the web, the resistance of two resistors; your site computes the total resistance of these resistors when placed in parallel, using the following formula:

$$\frac{1}{R_{total}} = \frac{1}{R_1} + \frac{1}{R_2}$$

The measured resistance of resistors, alas, comes with some error, thus you need to estimate the error in your calcluated total resistance. Estimate the error in your calculation when $R_1 = 500\Omega \pm 25\Omega$, and $R_2 = 3000\Omega \pm 100\Omega$. (If this doesn't work out for you, perhaps you might consider a career in cardboard box manufacturing...)

33. (§15.2) Given a function $f(x, y)$ and a rectangle R, evaluate

$$\iint\limits_R f(x,y)\,\mathrm{d}A$$

1. $f(x, y) = x + y, R = [0, 1] \times [2, 3]$.
2. $f(x, y) = x \sin y, R = [0, 1] \times [0, \pi]$.
3. $f(x, y) = x/y, R = [0, 1] \times [1, 2]$.
4. $f(x, y) = \cos(x + 2y), R = [0, \pi] \times [0, \pi/2]$.

34. (§15.3) Given a function $f(x, y)$ and a general region in the plane, D, evaluate

$$\iint\limits_D f(x,y)\,\mathrm{d}A$$

1. $f(x, y) = x - y$, D is the triangle with corners $(0, 0)$, $(0, 5)$, $(3, 2)$.
2. $f(x, y) = x + y, D = \left\{(x, y) \,\middle|\, x^2 + y^2 \le 4\right\}$. (actually this is easy by symmetry considerations.) (and is better done with polar coordinates anyway.)
3. $f(x, y) = \sqrt{1 - x^2}$, D is the triangle with corners $(0, 0)$, $(1, 0)$, $(1, 1)$.
4. $f(x, y) = (x - y)^2$, D is the region between the curves $y = x^2$ and $y = x$.

35. (§15.4) Given a function $f(x, y)$ and a general region in the plane, D, interpret the integral

$$\iint\limits_D f(x,y)\,\mathrm{d}A$$

as a polar integral and solve. *Note:* you should be able to recognize when it is appropriate or beneficial to evaluate a double integral in polar coordinates. *e.g.*,:

1. $f(x, y) = e^{x^2+y^2}, D = \left\{(x, y) \,\middle|\, x^2 + y^2 \le 9\right\}$.
2. $f(x, y) = 2, D = \left\{(x, y) \,\middle|\, x^2 + y^2 \le 1,\ y \ge x\right\}$.
3. $f(x, y) = x + y, D = \left\{(x, y) \,\middle|\, y \ge 0,\ 4 \le x^2 + y^2 \le 9\right\}$.
4. $f(x, y) = \sqrt{x^2 + y^2}, D = \{(r, \theta) \mid 0 \le \theta \le \pi/4,\ 0 \le r \le \cos 2\theta\}$.

36. (§15.5) Given a density function $\rho(x, y)$ for a lamina in the plane, D, find the center of mass of the lamina. *e.g.*,:

1. $\rho(x,y) = x + y + 2, D = \left\{(x,y) \,\middle|\, x^2 + y^2 \leq 1\right\}$.
2. $\rho(x,y) = 2, D = \left\{(x,y) \,\middle|\, 0 \leq x \leq 1,\, 0 \leq y \leq x^2\right\}$.
3. $\rho(x,y) = \frac{1}{\sqrt{x^2+y^2}}, D = \left\{(x,y) \,\middle|\, 0 \leq x,\, 1 \leq x^2 + y^2 \leq 4\right\}$. (*Hint:* try converting to polar integrals?)

37. (§15.7) Given a function $f(x,y,z)$, and some general region B in three space, evaluate

$$\iiint_B f(x,y,z)\,\mathrm{d}V$$

1. $f(x,y,z) = 1 + xy,\ B = \{(x,y,z) \mid 0 \leq x \leq 1,\ 0 \leq y \leq 2,\ 1 \leq z \leq 2\}$.
2. $f(x,y,z) = yz\cos\left(x^5\right),\ B = \{(x,y,z) \mid 0 \leq x \leq 1,\ 0 \leq y \leq x,\ x \leq z \leq 2x\}$. (This is §15.7 #8)
3. $f(x,y,z) = xz,\ B = \{(x,y,z) \mid 0 \leq x \leq 1,\ 0 \leq y \leq x,\ 0 \leq z \leq 6 - x - y\}$.

38. (§12.7) Convert the following points, given in cylindrical coordinates, to cartesian coordinates, and to spherical coordinates:

$$P = (1, \pi/3, 1) \qquad Q = (2, \pi, 0) \qquad R = \left(1, \pi/6, -\sqrt{3}\right)$$

39. (§12.7) Convert the following points, given in spherical coordinates, to cartesian coordinates, and to cylindrical coordinates:

$$P = [(\rho, \theta, \phi) = (1, \pi/2, \pi/3)] \qquad Q = [(\rho, \theta, \phi) = (2, 3\pi/2, \pi/2)]$$

40. (§15.8) Rewrite the triple integral

$$\iiint_R f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

in cylindrical coordinates. (**Note:** the resulting integrals might be very difficult or impossible.)

1. $f(x,y,z) = 3x^2+3y^2+7z$, and R is the region with $z \leq 5$, $x^2+y^2 \leq 4$, $x^2+y^2+z^2 \geq 4$.
2. $f(x,y,z) = 1$, and R is the region with $z \geq \sqrt{x^2+y^2}, x^2+y^2+z^2 \leq 32$. (Note: since you are integrating 1, the integral gives the volume of R.)
3. $f(x,y,z) = ze^{x^2+y^2}$, R is the region with $x \geq 0$, $y \geq 0$, $z \geq 0, z \leq 9 - x^2 - y^2$.

41. (§15.8) Rewrite the triple integral

$$\iiint_R f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

in spherical coordinates. (Again the resulting integrals might be difficult.)

1. $f(x,y,z) = 13\left(x^2+y^2+z^2\right)^{3/2}$, and R is the region $x^2+y^2+z^2 \leq 9$.
2. $f(x,y,z) = 5\left(x^2+y^2\right)$, and R is the region $x \geq 0, y \geq 0, z \geq 0, 4 \leq x^2+y^2+z^2 \leq 16$.

42. (misc) On the exam you may not be told what technique is appropriate for a given problem: you must decide this for yourself. In this category are the following miscellaneous problems:

1. Find the volume of the tetrahedron with corners $(0,0,0)$, $(2,0,0)$, $(0,1,0)$, $(0,0,3)$.
2. Find numbers a, b, c with $a+b+c=100$, such that a^2bc is a maximum.
3. Let $p(x,y,z) = x^2y + z^2x$ represent the density of plankton in the ocean. A whale is located at the point $(1,2,0)$. In which direction should the whale move to maximize the increase in plankton concentration?
4. Maximize $f(x,y) = 2x^3 + y^3$ subject to $x^2 + y^2 = 4$.