2005S M20C Final Exam Preparation

The final exam is Friday June 10, from 8:00am-11:00am. The exam *will be held in the Mandeville Auditorium,* in the Mandeville building. This is *not* our regular meeting place. You are warned. *You must bring a blue book to the exam.* Blue books are available from the bookstore. You also need to bring your student ID card or other form of ID (driver's license, passport, etc.) The exam is comprehensive: it covers all material from the quarter. There will be slightly more emphasis on material covered since the second midterm, that is §15.2–15.5, 15.7, 12.7, and whatever of §15.8 we get to in class on the last day.

You will have three hours to complete the final exam. It is, however, only about 50% longer than either midterm. So you should have plenty of time to complete the exam. The final exam is worth approximately 60% more than either of the midterms, thus it is based on 160 points. Thus one point on the final is about equal to one point on a midterm. A question which was worth about 15 points on midterm 1 would be worth the same on the final.

You may not use a calculator or notes during the exam.

The following formulæ will be provided on your exam:

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \qquad \boldsymbol{a} \times \boldsymbol{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

$$|\boldsymbol{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2} \qquad L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$$s = \int \left|\boldsymbol{R}'(t)\right| \mathrm{d}t \qquad \mathrm{D}_{\boldsymbol{u}} f = \nabla f \cdot \boldsymbol{u} \qquad \nabla f = \langle f_x, f_y \rangle \qquad D = f_{xx} f_{yy} - f_{yx} f_{xy}$$

$$\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta \qquad \mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z = \rho^2 \sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$$

Everything else must be committed to memory.

This exam preparation sheet is comprehensive. It includes the two midterm prep sheets as well as a prep sheet for material since the last midterm.

You should be prepared to answer at least the following questions:

1. What's the difference between a scalar and a vector? Describe the input and output types of common operations.

Answer: A vector comprises a direction and a scalar magnitude. A scalar can be conceived of as a vector in 1 dimension, and thus has trivial direction (*i.e.*, in the positive direction). The magnitude (or length) of a vector is a scalar. Scalar multiplication takes a scalar and a vector and produces a vector. The dot product takes two vectors and produces a scalar. The cross product takes two vectors and produces a third vector.

2. (§12.1) Let $P = (2,3,1)\,, Q = (4,-2,5)$.

1. What is the "position vector" of P?
2. Find the distance from P to Q.
3. Find the vector from P to Q, $\boldsymbol{PQ}$.
4. Find the distance from P to the origin.
5. Find the distance from P to the xy plane.
6. Find the equation of the sphere centered at P with radius 5.

7. Find the equation of the sphere centered at Q such that P is on the sphere.

Answer: 1. The vector $\langle 2,3,1\rangle$.

2. $\sqrt{45}$.
3. $\langle 2,-5,4\rangle$.
4. $\sqrt{14}$.
5. It is just the z coordinate, 1.
6. $(x-2)^2+(y-3)^2+(z-1)^2=25$.
7. $(x-4)^2+(y+2)^2+(z-5)^2=45$.

3. (§12.2–12.5) Let $\boldsymbol{a}=\langle 1,0,1\rangle$, $\boldsymbol{b}=\langle 1,3,0\rangle$.

1. Find $|\boldsymbol{a}|$.
2. Find a unit vector in the same direction as $\boldsymbol{a}$.
3. Find $\boldsymbol{a}+\boldsymbol{b}$.
4. Find $3\boldsymbol{a}-2\boldsymbol{b}$.
5. Find $\boldsymbol{a}\cdot\boldsymbol{b}$.
6. Find $\boldsymbol{a}\times\boldsymbol{b}$.
7. Find a vector perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$.
8. Find the angle subtended by the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.
9. Find the component of $\boldsymbol{a}$ along $\boldsymbol{b}$.
10. Find the projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$.
11. Give a parametrization of the line through the point $(3,4,1)$ parallel to $\boldsymbol{b}$.
12. Give an equation of the plane through the origin that is parallel to both $\boldsymbol{a}$ and $\boldsymbol{b}$.
13. Give an equation of the plane containing the point $(1,2,-1)$ that has $\boldsymbol{a}$ as a normal.
14. Find the angle subtended by $\boldsymbol{a}$ and $\boldsymbol{a}\times\boldsymbol{b}$.
15. Is $\boldsymbol{a}$ parallel to $\boldsymbol{b}$? Are they perpendicular?
16. Find a vector parallel to $\boldsymbol{a}$.

Answer: 1. $\sqrt{2}$.

2. $\left\langle \frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right\rangle$.
3. $\langle 2,3,1\rangle$.
4. $\langle 1,-6,3\rangle$.
5. 1.
6. $\langle -3,1,3\rangle$.
7. $\langle -3,1,3\rangle$.
8. $\arccos\left(\frac{1}{2\sqrt{5}}\right)$.
9. $\frac{1}{\sqrt{10}}$.
10. $\left\langle \frac{1}{10},\frac{3}{10},0\right\rangle$.
11. $\langle 3,4,1\rangle + t\,\langle 1,3,0\rangle$.
12. $-3x+y+3z=0$.

13. $x + z = 0$.
14. It should be $\pi/2$.
15. They are neither parallel nor perpendicular.
16. $\langle l, 0, l \rangle$, for any $l \neq 0$, is parallel to $\boldsymbol{a}$.

4. (§12.2–12.5) Let $\boldsymbol{u} = \langle 1, 2, -3 \rangle$, $\boldsymbol{v} = \langle -2, 4, 1 \rangle$, $\boldsymbol{w} = \langle -3, 0, 1 \rangle$.

1. Find $5\boldsymbol{u}$.
2. Find $\boldsymbol{u} \cdot \boldsymbol{u}$.
3. Find $|\boldsymbol{u}|$.
4. Find a unit length vector in the same direction as $\boldsymbol{u}$.
5. Find $|\alpha \boldsymbol{u}|$, where α is a real number.
6. Find $\boldsymbol{u} \cdot \boldsymbol{v}$.
7. Find the angle subtended by $\boldsymbol{u}$ and $\boldsymbol{v}$.
8. Find $\boldsymbol{v} \times \boldsymbol{w}$.
9. Find a vector perpendicular to both $\boldsymbol{v}$ and $\boldsymbol{w}$.
10. Find a unit length vector perpendicular to both $\boldsymbol{v}$ and $\boldsymbol{w}$.
11. Find $\boldsymbol{u} \times \boldsymbol{v}$.
12. Find $\boldsymbol{w} \cdot (\boldsymbol{u} \times \boldsymbol{v})$.
13. Find $\boldsymbol{u} \times \boldsymbol{u}$.

Answer:
1. $\langle 5, 10, -15 \rangle$
2. 14
3. $\sqrt{14}$
4. $(1/\sqrt{14})\boldsymbol{u} = \left\langle 1/\sqrt{14}, 2/\sqrt{14}, -3/\sqrt{14} \right\rangle$.
5. $|\alpha| \sqrt{14}$
6. $-2 + 8 - 3 = 3$
7. $\arccos\left(\boldsymbol{u} \cdot \boldsymbol{v} / |\boldsymbol{u}| \, |\boldsymbol{v}|\right) = \arccos\left(3/\sqrt{14}\sqrt{21}\right) \approx 80^\circ$
8. $\boldsymbol{v} \times \boldsymbol{w} = \langle 4, -1, 12 \rangle$
9. $\boldsymbol{v} \times \boldsymbol{w}$ is perpendicular to both $\boldsymbol{v}$ and $\boldsymbol{w}$.
10. $\boldsymbol{v} \times \boldsymbol{w} / |\boldsymbol{v} \times \boldsymbol{w}| = \langle 4, -1, 12 \rangle / \sqrt{161} = \left\langle 4/\sqrt{161}, -1/\sqrt{161}, 12/\sqrt{161} \right\rangle$.

5. (§12.5) Let $\boldsymbol{R_1}(t) = \langle 0, 3, 1 \rangle + t \langle 2, 1, -1 \rangle$ parametrize the line l_1. Let $\boldsymbol{R_2}(t) = \langle 1, 2, 1 \rangle + t \langle 2, 3, 0 \rangle$ parametrize the line l_2. Let $3x - 2y + 5z = 1$ describe a plane P.

1. Find a vector parallel to l_1. Find a vector parallel to l_2.
2. Find a vector normal to P.
3. Do the lines l_1, l_2 intersect? If so, find their point of intersection. If not, are they parallel, or are they skew?
4. Find the intersection of l_1 and P.
5. Find the intersection of l_2 and P. (*Hint:* it's a trick)
6. Find the angle that l_1 subtends with the plane P.
7. Find the distance from l_1 to the point $(2, 2, 1)$.

8. Find the distance from l_2 to P.

Answer: 1. $\langle 2,1,-1\rangle, \langle 2,3,0\rangle$.

2. $\langle 3,-2,5\rangle$.
3. They are skew.
4. $(-4,1,3)$.
5. They do not intersect.
6. $\pi/2 - \arccos\left(\frac{1}{2\sqrt{57}}\right) = \arcsin\left(\frac{1}{2\sqrt{57}}\right) \approx 3.8°$.
7. $\sqrt{7/2}$.
8. $3/\sqrt{38}$.

6. (§12.5) Let $3x - 2y + 5z = 1$ describe a plane P. Let $x + y + 3z = 4$ describe a plane Q.

1. Find a vector normal to P.
2. Find a vector normal to Q.
3. Are the planes P, Q parallel? If not, must they intersect?
4. Find the angle subtended by the two planes

Answer: 1. $\langle 3,-2,5\rangle$.

2. $\langle 1,1,3\rangle$.
3. Their normals are not parallel, so they are not parallel and must intersect.
4. Find the angle subtended by the two normals. It is

$$\arccos \frac{\langle 3,-2,5\rangle \cdot \langle 1,1,3\rangle}{|\langle 3,-2,5\rangle|\,|\langle 1,1,3\rangle|} = \arccos \frac{16}{\sqrt{38}\sqrt{11}}$$

7. (§13.1) Let $\boldsymbol{u} = \langle -2,1,4\rangle$, and let P be the point $(0,4,1)$.

1. Find a vector valued function $\boldsymbol{l}(t)$ which traces out the line through P and parallel to $\boldsymbol{u}$.
2. Find the equation of the plane containing P with normal $\boldsymbol{u}$.
3. Find the parametrization, $\boldsymbol{m}(t)$, of the line line through P and perpendicular to $\boldsymbol{u}$. (*Hint:* There are many answers. One way to do this is pick some arbitrary vector $\boldsymbol{v}$ not parallel to $\boldsymbol{u}$, then find a vector perpendicular to both $\boldsymbol{u}$ and $\boldsymbol{v}$.)

Answer: 1. One answer is $\boldsymbol{l}(t) = \langle 0,4,1\rangle + t\langle -2,1,4\rangle = \langle -2t, 4+t, 1+4t\rangle$.

2. Planes are of the form $ax+by+cz+d=0$, where $\langle a,b,c\rangle$ (and any scalar multiple of it) are normal to the plane. So the answer should be of the form $-2x+y+4z+d=0$. Plug in P as (x,y,z) to find the value of d. You should get $-2x+y+4z-8=0$.
3. First I will let $\boldsymbol{v} = \langle 0,0,1\rangle$, just for simplicity. Then I compute $\boldsymbol{w} = \boldsymbol{u}\times\boldsymbol{v} = \langle 1,2,0\rangle$, which is perpendicular to both $\boldsymbol{u}$ and $\boldsymbol{v}$. I only care that it is perpendicular to $\boldsymbol{u}$. If I had picked the "wrong" $\boldsymbol{v}$, this cross product would have been $\mathbf{0}$, and I would have known something was wrong and picked a different $\boldsymbol{v}$. Then I do as in the first part, getting $\boldsymbol{m}(t) = \langle 0,4,1\rangle + t\langle 1,2,0\rangle$.

8. (§13.2) Find the limits:

1. $\lim_{t\to 0} \boldsymbol{R}(t)$, where $\boldsymbol{R}(t) = \left\langle t+1, 5, t^2\right\rangle$.

2. $\lim_{t\to 0}\left\langle \frac{\sin t}{t}, e^{-t}, t\right\rangle$.
3. $\lim_{t\to 0}\left\langle \frac{t^2+t^4}{4t^2-t^3}, 5t+4, \ln(1+t)\right\rangle$.
4. $\lim_{t\to 1}\left\langle \frac{1-t}{1-t^2}, \frac{\ln t}{1-t}, 5t\right\rangle$.

Answer: 1. $\langle 1,5,0\rangle$.
2. $\langle 1,1,0\rangle$.
3. $\left\langle \frac{1}{4},4,0\right\rangle$.
4. $\left\langle \frac{1}{2},-1,5\right\rangle$.

9. (§13.2) Given $\boldsymbol{R}(t)$, find its antiderivative $\int \boldsymbol{R}(t)\,\mathrm{d}t$.
1. $\boldsymbol{R}(t) = \left\langle 2t-1, t^2, \cos t\right\rangle$.
2. $\boldsymbol{R}(t) = \left\langle \frac{1}{1+t^2}, \frac{1}{t}, \sqrt{2t+1}\right\rangle$.

Answer: 1. $\int \boldsymbol{R}(t)\,\mathrm{d}t = \left\langle t^2-t, t^3/3, \sin t\right\rangle + \boldsymbol{C}$
2. $\int \boldsymbol{R}(t)\,\mathrm{d}t = \left\langle \arctan t, \ln t, (2t+1)^{3/2}/3\right\rangle + \boldsymbol{C}$

10. (§13.2) Given vector valued functions $\boldsymbol{R}(t)$ and $\boldsymbol{S}(t)$, which describe the position of two particles, find whether the particles collide. Find if the paths of the two particles intersect.
1. $\boldsymbol{R}(t) = \left\langle 2t, t^2, \cos t\right\rangle$, $\boldsymbol{S}(t) = \left\langle \sin t, -e^t, 4t\right\rangle$.
2. $\boldsymbol{R}(t) = \langle 1+3t, 2-5t, -2+t\rangle$, $\boldsymbol{S}(t) = \langle 5-t, 1-4t, -3+2t\rangle$.
3. $\boldsymbol{R}(t) = \langle 1+t, 2+t, 3+t\rangle$, $\boldsymbol{S}(t) = \left\langle t-1, t^2, t+1\right\rangle$.

Answer: The particles collide if there is a t such that $\boldsymbol{R}(t) = \boldsymbol{S}(t)$. The paths intersect if there are t_1, t_2 such that $\boldsymbol{R}(t_1) = \boldsymbol{S}(t_2)$.

1. These particles cannot collide, and their paths cannot cross because the y component of $\boldsymbol{R}$ is nonnegative, while the y component of $\boldsymbol{S}$ is negative.
2. Try to solve $\langle 1+3t, 2-5t, -2+t\rangle = \langle 5-t, 1-4t, -3+2t\rangle$. This is three equations with one unknown, an overdetermined systems. Solving the first equation gives $1+3t = 5-t$, which has solution $t=1$. This value of t satisfies the other two equations, as you can verify:

$$\boldsymbol{R}(1) = \langle 4,-3,-1\rangle = \boldsymbol{S}(1).$$

3. First try to solve $\langle 1+t, 2+t, 3+t\rangle = \left\langle t-1, t^2, t+1\right\rangle$. This is three equations with one unknown. Solving the first equation gives $1+t = t-1$. This has no solution. Thus we should look for path intersection. So we try to solve $\langle 1+t_1, 2+t_1, 3+t_1\rangle = \left\langle t_2-1, t_2^2, t_2+1\right\rangle$. This is three equations with two unknowns, which is still overdetermined, but may have a solution Solving the first equation gives $1+t_1 = t_2-1$, or $t_1+2 = t_2$. Plugging this into the second equation gives $2+t_1 = t_2^2 = (t_1+2)^2 = t_1^2+4t_1+4$. This is the quadratic equation $t_1^2+3t_1+2=0$, with solutions $t_1 = -1, -2$. This gives prospective solutions $t_1=-1, t_2=1$, and $t_1=-2, t_2=0$. Check if these satisfy the third equation $3+t_1 = t_2+1$. They both satisfy it. So we get two points of intersection:

$$\boldsymbol{R}(-1) = \langle 0,1,2\rangle = \boldsymbol{S}(1), \quad \text{and} \quad \boldsymbol{R}(-2) = \langle -1,0,1\rangle = \boldsymbol{S}(0).$$

11. (§13.2) Find $\boldsymbol{R}'(t)$ for

1. $\boldsymbol{R}(t) = \langle t, 3-2t, 4+6t\rangle$.
2. $\boldsymbol{R}(t) = \left\langle t+1, 5, t^2\right\rangle$.
3. $\boldsymbol{R}(t) = \langle 4,7,1\rangle \times \left\langle t+1, 5, t^2\right\rangle$.
4. $\boldsymbol{R}(t) = \langle \arctan t, \sin t, \cos t\rangle$.
5. $\boldsymbol{R}(t) = \left\langle \arctan\left(e^t\right), \sin\left(e^t\right), \cos\left(e^t\right)\right\rangle$.
6. $\boldsymbol{R}(t) = \langle \cos t, \sin t, 3t\rangle$.
7. $\boldsymbol{R}(t) = \left\langle (t+1)\sin t, 5\sin t, t^2 \sin t\right\rangle$.

Answer: 1. $\boldsymbol{R}'(t) = \langle 1,-2,6\rangle$.

2. $\boldsymbol{R}'(t) = \langle 1, 0, 2t\rangle$.
3. $\boldsymbol{R}'(t) = \langle 4,7,1\rangle \times \langle 1,0,2t\rangle = \langle 14t, 1-8t, -7\rangle$.
4. $\boldsymbol{R}'(t) = \left\langle \frac{1}{1+t^2}, \cos t, -\sin t\right\rangle$.
5. $\boldsymbol{R}'(t) = \langle -\sin t, \cos t, 3\rangle$.
6. $\boldsymbol{R}'(t) = \sin t\,\langle 1,0,2t\rangle + \cos t\left\langle t+1,5,t^2\right\rangle$

12. (§13.3) Given vector valued function $\boldsymbol{R}(t)$, find the arc length of the curve traced by $\boldsymbol{R}$ for t between t_0 and t_1, *e.g.*,

1. $\boldsymbol{R}(t) = \langle 1+3t, 3, 1-4t\rangle$, $t_0 = 0,\ t_1 = 3$.
2. $\boldsymbol{R}(t) = \langle 2\cos t, 2\sin t\rangle, 0 \le t \le \pi$.
3. $\boldsymbol{R}(t) = \langle \cos t, \sin t, 3t\rangle$, $t_0 = 0,\ t_1 = 4\pi$.
4. $\boldsymbol{R}(t) = \left\langle t/\sqrt{2}, t/\sqrt{2}, \ln(\sec t)\right\rangle$, $t_0 = 0,\ t_1 = \pi/3$.
5. $\boldsymbol{R}(t) = \left\langle t, t^3/3 + 1/(4t)\right\rangle$, $t_0 = 1,\ t_1 = 2$.
6. $\boldsymbol{R}(t) = \left\langle \sqrt{2}t, e^t, e^{-t}\right\rangle, 0 \le t \le 1$.
7. $\boldsymbol{R}(t) = \left\langle t^2, t, 3\right\rangle, 0 \le t \le 1$.

Answer: Solveable problems of this type are few and far between. These examples pretty much exhaust my supply of good problems. Recall that the arc length of $\boldsymbol{c}(t)$ for $t_0 \le t \le t_1$ is

$$\int_{t_0}^{t_1} \left|\boldsymbol{c}'(t)\right|\,\mathrm{d}t$$

1. $\int_0^3 5\,\mathrm{d}t = 15$
2. 2π.
3. $\int_0^{4\pi} \sqrt{10}\,\mathrm{d}t = 4\pi\sqrt{10}$.
4. $\int_0^{\pi/3} \sec t\,\mathrm{d}t = \ln\left(\sec t + \cos t\right)\Big|_0^{\pi/3} = 2 + \sqrt{3} - 1 = 1 + \sqrt{3}$.

5. We compute $\boldsymbol{R}'(t) = \left\langle 1, t^2 - 1/(4t^2) \right\rangle$. Now note that

$$\begin{aligned} \left|\boldsymbol{R}'(t)\right| &= \sqrt{1 + (t^2 - 1/(4t^2))^2} = \sqrt{1 + t^4 - 1/2 + 1/(16t^4)} \\ &= \sqrt{t^4 + 1/2 + 1/(16t^4)} = \sqrt{(t^2 + 1/(4t^2))^2} = t^2 + 1/(4t^2) \end{aligned}$$

Thus the answer is

$$\int_1^2 t^2 + 1/(4t^2)\,\mathrm{d}t = t^3/3 - 1/(4t)\Big|_1^2 = 8/3 - 1/8 - (1/3 - 1/4) = 7/3 + 1/8$$

6. $e^1 - e^{-1}$.
7. The integral might be tricky. I get $\frac{\sqrt{5}}{2} + \frac{1}{4}\operatorname{arcsinh} 2$.

13. (§13.4) Given the position of a particle, $\boldsymbol{R}(t)$, find its velocity, speed, and acceleration. *e.g.,*:

1. $\boldsymbol{R}(t) = \langle 2\cos t, 2\sin t \rangle$.
2. $\boldsymbol{R}(t) = \left\langle \sqrt{2}t, e^t, e^{-t} \right\rangle$.
3. $\boldsymbol{R}(t) = \left\langle t^2, t, 3 \right\rangle$.

Answer: 1. $\boldsymbol{v}(t) = \langle -2\sin t, 2\cos t \rangle$, $s(t) = 2$, $\boldsymbol{a}(t) = \langle -2\cos t, -2\sin t \rangle$.
2. $\boldsymbol{v}(t) = \left\langle \sqrt{2}, e^t, -e^{-t} \right\rangle$, $s(t) = e^t + e^{-t}$, $\boldsymbol{a}(t) = \left\langle 0, e^t, e^{-t} \right\rangle$.
3. $\boldsymbol{v}(t) = \langle 2t, 1, 0 \rangle$, $s(t) = \sqrt{4t^2 + 1}$, $\boldsymbol{a}(t) = \langle 2, 0, 0 \rangle$.

14. (§13.4) Given the acceleration of a particle and its initial position and velocity, find the position of the particle. *e.g.,*:

1. $\boldsymbol{a}(t) = \langle 1, t, \sin t \rangle$, $\boldsymbol{v}(0) = \langle 0, 1, 0 \rangle$, $\boldsymbol{R}(0) = \langle 0, 0, 0 \rangle$.
2. $\boldsymbol{a}(t) = \langle \sin t, \cos t \rangle$, $\boldsymbol{v}(0) = \langle 1, 0 \rangle$, $\boldsymbol{R}(0) = \langle 0, 1 \rangle$.

Answer: 1. $\boldsymbol{R}(t) = \left\langle \frac{1}{2}t^2, \frac{1}{6}t^3 + t, t - \sin t \right\rangle$.
2. $\boldsymbol{R}(t) = \langle 2t - \sin t, 2 - \cos t \rangle$.

15. (§14.1) You should have some basic understanding of functions of two or more variables. *e.g.,*: given $f(x, y)$, be able to evaluate $f(1, 2)$; find the domain of f; sketch a graph of $f(x, y)$, or the contours (level sets) of $f(x, y)$.

1. What is the domain of $f(x, y) = \frac{\sqrt{x-y}}{y}$?
2. Graph the level sets of $f(x, y) = \frac{1}{4}x^2 + y^2$.
3. Graph the function $f(x, y) = \sqrt{x^2 + y^2}$.
4. Graph the function and level sets of $f(x, y) = 3x - 4y + 2$.

Answer: 1. The points (x, y) with $x \geq y$, and $y \neq 0$.
2. These are ellipses.
3. A paraboloid.
4. The graph is the plane $-3x + 4y + z = 2$, the level sets are lines.

16. (§14.1) (this is a basic one) You should understand functions of two or more variables. *e.g.,*: given $f(x, y)$, be able to evaluate $f(1, 2)$, or $f(\pi, 100)$; find the domain of f; sketch a graph of $f(x, y)$, or the contours (level sets) of $f(x, y)$.

1. What is the domain of $f(x,y) = \frac{\sqrt{x-y}}{y}$?
2. What is the domain of $f(x,y) = \frac{xy^3}{x^2+y^2}$?
3. Graph the level sets of $f(x,y) = \frac{1}{4}x^2 + y^2$.
4. Graph the function $f(x,y) = \sqrt{x^2+y^2}$.

Answer: 1. The points (x,y) with $x \geq y$, and $y \neq 0$.

2. The points (x,y) with $x \neq 0$, and $y \neq 0$. Sometimes written as $\mathbb{R}^2 \setminus \{(0,0)\}$, that is all the real plane, $\mathbb{R}^2$, without the single point $(0,0)$.
3. These are ellipses.
4. A paraboloid.

17. (§14.1) Sketch the level sets of $f(x,y)$ for various values of k

1. $f(x,y) = x^2 + y^2$
2. $f(x,y) = 2x - 4y + 7$
3. $f(x,y) = \sqrt{x^2+y^2}$

Answer: 1. level set of value k is a circle of radius $\sqrt{k}$ centered at the origin.

2. level set of value k is a line with slope $\frac{1}{2}$ and y-intercept of $(7-k)/4$
3. level set of value k is a circle of radius k centered at the origin.

18. (§14.2) Identify if a function $f(x,y)$ is continuous at a given point (x,y). Identify if the function is continuous for all points in $\mathbb{R}^2$.

1. $f(x,y) = x^2 + 4x^3y^2$ at $(x,y) = (1,3)$.
2. $f(x,y) = \begin{cases} \frac{x+y}{x^2+y^2} & (x,y) \neq (0,0) \\ 1 & (x,y) = (0,0) \end{cases}$ at $(x,y) = (0,0)$.
3. $f(x,y) = \frac{x^3+y^3}{\sqrt{x+y}}$ at $(x,y) = (2,1)$.

Answer: 1. This function is continuous for all points in $\mathbb{R}^2$, so it is continuous at $(1,3)$.

2. The function is *not* continuous at $(0,0)$, but is continuous at all other points. To show this you should consider the limit of $f(x,y)$ as $(x,y) \to (0,0)$; this limit is not defined, thus f is not continuous. However, if you wanted to somehow use continuity of f to *compute* the limit of f as $(x,y) \to (0,0)$, you would have a "chicken and the egg" problem.
   This illustrates the general principle: if you suspect the limit $\lim_{(x,y)\to(a,b)} f(x,y)$ exists, it may be easiest to show the function is continuous at (a,b), then you only need evaluate $f(a,b)$. However, if the function is not continuous at the point, or you think the limit does not exist, this is not the way to go. To show the limit does not exist, it may be better to find two paths approaching (a,b) which give different limits of f.
3. This function is continuous for all points such that $x+y > 0$, thus it is continuous at $(2,1)$.

19. (§14.2) You should be able to find the limit of a function of two variables, or recognize that the limit does not exist. You need to know what functions are continuous, and how this helps you evaluate a limit. *e.g.*, evaluate:

1. $\lim_{(x,y)\to(1,1)} \frac{x^6 y}{2x^2-y}$.
2. $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{4+x^2+\cos y}$,
3. $\lim_{(x,y)\to(0,0)} \frac{x-y}{x+y}$,
4. $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$,
5. $\lim_{(x,y)\to(0,0)} \frac{6x^3 y}{2x^4+y^4}$.
6. $\lim_{(x,y)\to(0,0)} \frac{xy+x^2y^2}{xy^2+x^2y}$.

Answer: 1. By continuity, this is 1.

2. By continuity, this is 0.
3. Does not exist.
4. Does not exist.
5. Does not exist.
6. This one has some domain problems. First try $x = y > 0$, with $x \to 0$. Then try $x = y < 0$ with $x \to 0$. You should get $\infty, -\infty$ respectively. Thus the limit does not exist. This is a considerably harder problem than the rest of them.

20. (§14.3) Given a function $f(x, y)$, find the partials $f_x, f_y, f_{xx}, f_{xy}, f_{yy}$, etc. When do we have $f_{xy} = f_{yx}$?

1. $f(x, y) = 3x + 2y + 4$
2. $f(x, y) = 5x^2 + 2xy - 3y^2 + x - y$
3. $f(x, y) = e^{xy}$
4. $f(x, y) = x^2 \sin xy^2$

Answer: Clairaut's theorem states that if the second partials f_{xy} and f_{yx} exist and are continuous, then they are equal.

1. $f_x = 3$, $f_y = 2$, $f_{xx} = 0$, $f_{yy} = 0$, $f_{xy} = f_{yx} = 0$.
2. $f_x = 10x + 2y + 1$, $f_y = 2x - 6y - 1$, $f_{xx} = 10$, $f_{yy} = -6$, $f_{xy} = f_{yx} = 2$.
3. $f_x = ye^{xy}$, $f_y = xe^{xy}$, $f_{xx} = y^2 e^{xy}$, $f_{yy} = x^2 e^{xy}$, $f_{xy} = f_{yx} = yxe^{xy} + e^{xy}$.
4. $f_x = 2x \sin xy^2 + x^2 y^2 \cos xy^2$, $f_y = 2x^3 y \cos xy^2$, $f_{xx} = 2 \sin xy^2 + 4xy^2 \cos xy^2 - x^2 y^4 \sin xy^2$, $f_{yy} = 2x^3 \cos xy^2 - 4x^4 y^2 \sin xy^2$, $f_{xy} = f_{yx} = 6x^2 y \cos xy^2 - 2x^3 y^3 \sin xy^2$.

21. (§14.4) Given a function $f(x, y)$, and point (x_0, y_0), find the equation of the tangent plane to f at this point.

1. $f(x, y) = x^2 + 2xy + y^2, (x_0, y_0) = (1, 1)$.
2. $f(x, y) = \cos\left(xy^2\right), (x_0, y_0) = (\pi, 1)$.
3. $f(x, y) = e^{xy}, (x_0, y_0) = (2, 3)$.
4. $f(x, y) = \sqrt{x^2 + y^2}, (x_0, y_0) = (-3, 4)$.

Answer: 1. $z - 4 = 4(x - 1) + 4(y - 1)$.

2. $z = -1$.

3. $z - e^6 = 3e^6(x-2) + 2e^6(y-3)$.
4. $z - 5 = (-3/5)(x+3) + (4/5)(y-4)$.

22. (§14.4) Given a function $f(x,y)$, and point (x_0, y_0), find the equation of the linearization of f, call it $L(x,y)$ at the given point. Use the linearization to approximate $f(x,y)$ for some point $(x,y) \approx (x_0, y_0)$.

1. $f(x,y) = (x+y)^3$, $(x_0, y_0) = (1,1)$, approximate $f(0.9, 1.1)$.
2. $f(x,y) = \sqrt{xy}$, $(x_0, y_0) = (2,8)$, approximate $f(2.1, 8.1)$.
3. Approximate $\sin(\pi + 0.1)\cos(0.05)$.

Answer: The linearization has the form

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$$

1. $L(x,y) = 8 + 12(x-1) + 12(y-1)$. $L(0.9, 1.1) = 8$.
2. $L(x,y) = 4 + (x-2) + \frac{1}{4}(y-8) = x + \frac{1}{4}y$. $L(2.1, 8.1) = 4.125$.
3. We are letting $f(x,y) = \sin(x)\cos(y)$. We can easily find the Linearization expanded around $(\pi, 0)$: $L(x,y) = f(\pi,0) + \nabla f(\pi, 0) \cdot \langle x - \pi, y\rangle$. This gives $L(x,y) = 0 + \langle -1, 0\rangle \cdot \langle x - \pi, y\rangle = \pi - x$. Then we have $f(\pi + 0.1, 0.05) \approx L(\pi + 0.1, 0.05) = \pi - (\pi + 0.1) = -0.1$. The real answer is $-0.0997...$

23. (§14.5) Be prepared to use the chain rule to find derivatives and partial derivatives. *e.g.*,:

1. $f(x,y) = (x+y)^4$, $x = \sqrt{t}, y = t^2$, find $\frac{\mathrm{d}f}{\mathrm{d}t}$.
2. $f(x,y) = x^2 + \cos(xy)$, $x = s + t, y = s^2$, find $\frac{\partial f}{\partial s}$, and $\frac{\partial f}{\partial t}$.
3. $f(x,y,z) = \sqrt{x^2 + y^2 + z^2}$, $x = s\cos t, y = s\sin t, z = s^2$, find $\frac{\partial f}{\partial s}$, and $\frac{\partial f}{\partial t}$.
4. $f(x,y,z) = \left(x^a + y^b + z^c\right)^n$, $x = r\sin s\cos t, y = r\sin s\sin s, z = r\cos s$. Find $\frac{\partial f}{\partial r}$, $\frac{\partial f}{\partial s}$, and $\frac{\partial f}{\partial t}$.

Answer:

1. $4\left(\sqrt{t} + t^2\right)\left[\frac{1}{2\sqrt{t}} + 2t\right]$.
2. $\frac{\partial f}{\partial s} = \nabla f \cdot \langle 1, 2s\rangle$, $\frac{\partial f}{\partial t} = \nabla f \cdot \langle 1, 0\rangle$.
3. $\frac{\partial f}{\partial s} = \frac{1}{\sqrt{s^2+s^4}}\left(s + 2s^3\right)$, $\frac{\partial f}{\partial t} = 0$.

24. (§14.6) Given a function $f(x,y)$, or $f(x,y,z)$, find the gradient ∇f.

1. $f(x,y) = x^2 y + y$.
2. $f(x,y) = \cos(xy) + x^2 + y^2$.
3. $f(x,y) = \sqrt{x^2 + y^2}$.
4. $f(x,y) = \left(x^a + y^b\right)^n$.
5. $f(x,y,z) = \sin(z)\, e^{xy}$.
6. $f(x,y,z) = \sqrt{x^2 + y^2 - z^2}$.
7. $f(x,y,z) = \sin(xy) + \cos(yz)$.
8. $f(x,y) = \frac{xy(x^2-y^2)}{x^2+y^2}$.

Answer:

1. $\nabla f = \left\langle 2xy, x^2 + 1\right\rangle$.
2. $\nabla f = \langle -\sin(xy)\, y + 2x, -\sin(xy)\, x + 2y\rangle$.

3. $\nabla f = \left\langle x/\sqrt{x^2+y^2}, y/\sqrt{x^2+y^2}\right\rangle = (1/f)\langle x, y\rangle$.
4. $\nabla f = \left\langle nax^{a-1}\left(x^a+y^b\right)^{n-1}, nby^{b-1}\left(x^a+y^b\right)^{n-1}\right\rangle$.
5. $\nabla f = \langle \sin(z)\, ye^{xy}, \sin(z)\, xe^{xy}, \cos(z)\, e^{xy}\rangle = f(x,y,z)\langle y, x, \cot z\rangle$.
6. $\nabla f = \left\langle x/\sqrt{x^2+y^2-z^2}, y/\sqrt{x^2+y^2-z^2}, -z/\sqrt{x^2+y^2-z^2}\right\rangle = (1/f(x,y,z))\langle x, y, -z\rangle$.
7. $\nabla f = \langle y\cos(xy), x\cos(xy) - z\sin(yz), -y\sin(yz)\rangle$

25. (§14.6) Given a function $f(x,y)$, and a unit vector $\boldsymbol{u}$, find the directional derivative of f in the direction of $\boldsymbol{u}$, $\mathrm{D}_{\boldsymbol{u}}f(x,y)$. *e.g.*,:

1. $f(x,y) = x^2 + xy + y^2$, $\boldsymbol{u} = \left\langle \frac{3}{5}, \frac{4}{5}\right\rangle$.
2. $f(x,y) = \cos\frac{x}{y} + \cos\frac{y}{x}$, $\boldsymbol{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle$.
3. $f(x,y) = \arctan xy$, $\boldsymbol{u} = \langle 1, 0\rangle$, at the point $(x,y) = (1,1)$.
4. $f(x,y) = \sqrt{x^2+y^2}$, $\boldsymbol{u} = \langle \cos\theta, \sin\theta\rangle$, for some θ.

Answer: Remember that $\mathrm{D}_{\boldsymbol{u}}f(x,y) = \nabla f(x,y)\cdot\boldsymbol{u}$. This will be one of the formulæ provided on the top of your exam sheet.

1. $2x + \frac{11}{5}y$.
2. $\frac{1}{\sqrt{2}}\left(-\frac{1}{y}\sin\frac{x}{y} + \frac{y}{x^2}\sin\frac{y}{x} - \frac{1}{x}\sin\frac{y}{x} + \frac{x}{y^2}\sin\frac{x}{y}\right)$.
3. $\frac{1}{2}$.
4. $\nabla f(x,y)\cdot\langle\cos\theta, \sin\theta\rangle = (1/f(x,y))\langle x, y\rangle\cdot\langle\cos\theta,\sin\theta\rangle = (x\cos\theta + y\sin\theta)/\sqrt{x^2+y^2}$.

26. (§14.6) Understand the connection between ∇f and the directional derivative, the direction of maximal increase of f, and the maximal rate of instantaneous increase of f. *e.g.*,:

1. For $f(x,y) = \sqrt{x^2+y^2}$ at $(3,4)$, what is the direction and rate of maximal increase of f?
2. Let $f(x,y) = \sin\left(x^2y\right) + y^2$ denote the elevation of a mountain at point (x,y). A hiker is at $(0, \sqrt{\pi})$, and wishes to identify the direction of steepest ascent up the mountain. Which direction is that?
3. Captain Queeg's submarine is in trouble off the coast of Madagascar: radioactive lava is flowing out of a vent in the sea floor. Under a local coordinate system, the concentration of radioactive stuff is given as

$$f(x,y,z) = \frac{1}{1+(x-2)^2+(y+1)^2+(z-3)^2}.$$

Captain Queeg's submarine is located at $(1,1,2)$ under this coordinate system, and the crew is quickly being sickened from the effects of radiation. In which direction should the Captain steer the ship to experience the quickest (i.e. maximal) decrease in concentration of radioactivity?

Answer: The direction of maximal increase is a unit vector in the direction of ∇f. The maximal rate of instantaneous increase is $|\nabla f|$.

1. $\langle 3/5, 4/5\rangle$, and the rate is 1.
2. Direction of steepest ascent is $\nabla f = \left\langle 2xy\cos\left(x^2y\right), x^2\cos\left(x^2y\right) + 2y\right\rangle$. At the point

$(0,\sqrt{\pi})$ this is $\nabla f\left(0,\sqrt{\pi}\right) = \langle 0, 2\sqrt{\pi}\rangle$. Thus the direction of steepest ascent is the unit vector in this direction, $\langle 0, 1\rangle$.

3. the direction of maximal decrease in f is the unit vector in direction $-\nabla f$. Thus we find

$$\nabla f\left(x,y,z\right) = -\left(f\left(x,y,z\right)\right)^2 \langle 2(x-2), 2(y+1), 2(z-3)\rangle.$$

This gives $\nabla f\left(1,1,2\right) = -\frac{1}{(1+(-1)^2+2^2+(-1)^2)^2}\langle 2(-1), 2(2), 2(-1)\rangle = (2/49)\langle -1, 2, -1\rangle$. The unit vector in the same direction is $\left(1/\sqrt{6}\right)\langle -1, 2, -1\rangle$.

27. (§14.6) Understand the connection between ∇f and a level set of f. *e.g.,*:

1. Find the equation of the plane tangent to the surface $x^2 + y^2 + z^2 = 16$ at the point $\left(2, 2\sqrt{2}, 2\right)$.
2. Find the equation of the plane tangent to the surface $x^2 + y^2 - z = 9$ at the point $(3, 2, 4)$.
3. Find the equation of the plane tangent to the surface $a^2x^2 + b^2y^2 + c^2z^2 = 3r^2$ at the point $(r/a, r/b, r/c)$.

Answer: View the surface as a level set $f(x,y,z) = k$, then find gradient $\nabla f(x_0, y_0, z_0)$. This vector is the plane's normal. Plug in the point to get the equation of the plane.

1. $4x + 4\sqrt{2}y + 4z - 32 = 0$
2. $6x + 4y - z - 22 = 0$
3. $2arx + 2bry + 2crz - 6r^2 = 0$, or, more simply, $ax + by + cz = 3r$.

28. (§14.7) Given a function $f(x,y)$, be prepared to find all its critical points. Be able to apply the second derivative test to see if you have a max, min, or neither. Be able to find extremal values of $f(x,y)$ on a closed, bounded domain D. *e.g.,*:

1. $f(x,y) = x^4 + y^4 - 4xy$.
2. $f(x,y) = (1 + xy)(x + y)$.
3. $f(x,y) = e^x \cos y$.
4. $f(x,y) = x + 3y - 4$ on the triangle with corners $(0,0)$,$(1,0)$,$(0,1)$.
5. $f(x,y) = 2x^2 + 3y^2 - 4x - 5$ on the disc $x^2 + y^2 \le 16$.

Answer: A critical point is a point where ∇f either does not exist or is equal to $\mathbf{0}$. In the latter case we can use the second derivative test, if the second derivatives exist and are continuous, then we can evaluate $D = f_{xx}f_{yy} - f_{xy}^2$. If this quantity is positive, then we have a max or min. If it is negative, we have a saddle point. If the quantity is positive, we go on to check the sign of f_{xx}. If it is positive, we have a min, otherwise a max.

1. Local minima at $(-1,-1)$ and $(1,1)$. Saddle point at $(0,0)$.
2. $f(x,y) = x + y + x^2y + xy^2$, so $\nabla f = \left\langle 1 + 2xy + y^2, 1 + 2xy + x^2\right\rangle$. The critical points occur where $\nabla f = \mathbf{0}$. This gives two equations in two unknowns:

$$1 + 2xy + y^2 = 0 \quad \text{and} \quad 1 + 2xy + x^2 = 0.$$

Set both right hand sides equal to each other: $1 + 2xy + y^2 = 1 + 2xy + x^2$, and subtract to get $y^2 = x^2$, which is to say $y = \pm x$. Plug this back into the original equation:

$$1 + 2x(\pm x) + x^2 = 0$$

This gives two quadratic equations in x depending on whether $y = +x$ or $y = -x$. That is, we get

$$1 + 3x^2 = 0 \quad \text{and} \quad 1 - x^2 = 0$$

The former equation has only imaginary roots, so this leaves the latter, which has roots $x = \pm 1$. This gives us critical points $(1, -1)$ and $(-1, 1)$.
The discriminant is $D = 4xy - (2x + 2y)^2 = 4xy - 4x^2 - 8xy - 4y^2$. This is negative for both critical points, thus they are saddle points both.

3. This has no critical points. Thus no extrema on the unbounded set $\mathbb{R}^2$.
4. This has no critical points. Min of -4 at $(0, 0)$, and max of -1 at $(0, 1)$.
5. I did this one in class.

29. (§14.8) Given a function $f(x, y)$, find the extremal values of f subject to the constraint $g(x, y) = k$ by the method of Lagrange Multipliers.

1. $f(x, y) = x + xy + y$, subject to $x^2 + y^2 = 9$.
2. $f(x, y) = \frac{1}{xy}$, subject to $x + y = 4$.
3. $f(x, y, z) = 8x - 4z$, subject to $x^2 + 10y^2 + z^2 = 5$.
4. $f(x, y, z) = xyz$, subject to $2xz + 2yz + xy = 12$.

Answer: Lagrange's method: Find x, y, λ such that $\nabla f(x, y) = \lambda g(x, y)$ and $g(x, y) = k$. After finding all such (x, y), tabulate your results and compare the values of f.

1. At $\left(-\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right)$, takes min value $\frac{9}{2} - \frac{6}{\sqrt{2}}$. At $\left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$, takes max value $\frac{9}{2} + \frac{6}{\sqrt{2}}$.
2. This is a poor Lagrange question, but we expect a local min of $\frac{1}{4}$ at $(2, 2)$.
3. At $(-2, 1)$ takes min value -20. At $(2, -1)$ takes max value 20
4. I did something like this in class.

30. (§15.1) Given a function $f(x, y)$ and a rectangle $R = [a, b] \times [c, d]$, define the integral

$$\iint_R f(x, y)\, \mathrm{d}A$$

Answer: The integral is the limit of the Riemann sums

$$\lim_{m\to\infty,\, n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f\left(x^*_{i,j}, y^*_{i,j}\right) \Delta x \Delta y,$$

where $\Delta x = (b - a)/m$, and $\Delta y = (d - c)/n$, and

$$a+(i-1)\Delta x = x_{i-1} \le x^*_{i,j} \le x_i = a+i\Delta x \quad \text{and} \quad c+(j-1)\Delta y = y_{j-1} \le y^*_{i,j} \le y_i = c+j\Delta y.$$

31. (§15.1) Estimate the integral $\iint_R f(x, y)\, \mathrm{d}A$ by a Riemann Sum.

1. Do this for $f(x,y) = x^2 - xy^2$ for $R = [1,2] \times [1,4]$ with $m = 2, n = 3$. Take $\left(x^*_{i,j}, y^*_{i,j}\right)$ to be the lower left corner of each rectangle.
2. Do this for $f(x,y) = x^2 - 2xy + y^2$ for $R = [-1,1] \times [-1,1]$ with $m = 2, n = 2$. Take $\left(x^*_{i,j}, y^*_{i,j}\right)$ to be the midpoint of each rectangle.

Answer: 1. With $\Delta x = (2-1)/2 = 1/2$, $\Delta y = (4-1)/3 = 1$, we have $x_0 = 1, x_1 = 3/2, x_2 = 2$, and $y_0 = 1, y_1 = 2, y_2 = 3, y_3 = 4$. Then our Riemann Sum is

$$\begin{aligned}
&\sum_{i=1}^{2}\sum_{j=1}^{3} f\left(x^*_{i,j}, y^*_{i,j}\right)(1/2)(1) \\
&= \frac{1}{2}\left[f(1,1) + f(1,2) + f(1,3) + f(3/2,1) + f(3/2,2) + f(3/2,3)\right] \\
&= \frac{1}{2}\left[(1-1) + (1-4) + (1-9) + (9/4-3/2) + (9/4-6) + (9/4-27/2)\right] \\
&= \frac{1}{2}\left[0 - 3 - 8 + 27/4 - 6 - 30/2\right] = \frac{1}{2}\left[-11 + 6 + 3/4 - 6 - 15\right] = -101/8
\end{aligned}$$

2. The answer is 2.

32. (misc.) 1. On your first day working in the cardboard box factory, you are given the problem of minimizing the cost of your company's next generation product: the cardboard box. This cardboard box of the future consists of a bottom, four sides and a top, all of which are rectangular. The bottom is made of a special cardboard, called KardBord MIII^{TM}, which costs 5 cents per square inch. The sides and the top are made out of second generation recycled cardboard, called ReSyKleBoRd 2^{TM}, which costs 3 cents per square inch. The total volume of the box is to be 972 cubic inches. Due to restrictions beyond your control (*i.e.*, dictated by UPS and FedEx), the width of the box may not exceed 7 inches. Find the minimal cost size box. Your next assignment: find a better job than working in a cardboard factory.

2. Let a, b, c be positive integers. Find positive real numbers x, y, z such that $x+y+z = 117$ and such that $x^a y^b z^c$ is maximized.
3. You are the new chief operating officer at `www.ohmcalc.com`, the newest way to calculate resistance over the internet. Registered (*i.e.*, paying) users have the capability of entering, via the web, the resistance of two resistors; your site computes the total resistance of these resistors when placed in parallel, using the following formula:

$$\frac{1}{R_{total}} = \frac{1}{R_1} + \frac{1}{R_2}$$

The measured resistance of resistors, alas, comes with some error, thus you need to estimate the error in your calcluated total resistance. Estimate the error in your calculation when $R_1 = 500\Omega \pm 25\Omega$, and $R_2 = 3000\Omega \pm 100\Omega$. (If this doesn't work out for you, perhaps you might consider a career in cardboard box manufacturing...)

Answer: 1. Let x, y be the width and the length, and let $972/xy$ be the height. We wish to minimize

$$f(x,y) = 5xy + 3\left[2(x972/xy) + 2(y972/xy) + xy\right] = 8xy + 5832\left[\frac{1}{y} + \frac{1}{x}\right]$$

We also have the restrictions $x \in (0, 7]$, and $y \in (0, \infty)$
We find the gradient

$$\nabla f(x, y) = \left\langle 8y - 5832/x^2, 8x - 5832/y^2 \right\rangle$$

and set equal to $\mathbf{0}$. This gives two equations, which we set equal to each other:

$$\begin{aligned} 8y - 5832/x^2 &= 0 = 8x - 5832/y^2, \quad \text{so} \\ 8x^2y^3 - 5832y^2 &= 8x^3y^2 - 5832x^2, \quad \text{or} \\ 0 &= 8x^2y^2(x - y) - 5832(x^2 - y^2) \\ 0 &= 8x^2y^2(x - y) - 5832(x + y)(x - y) = (x - y)\left[8x^2y^2 - 5832(x + y)\right] \end{aligned}$$

This gives two possibilities: $x - y = 0$ or $x^2y^2 = 729(x + y)$. The latter condition cannot be satisfied, as we have

$$\begin{aligned} 0 &= 8y - 5832/x^2, \quad \text{so} \\ y &= 729/x^2, \\ y^2x^2 &= 729y, \end{aligned}$$

Subtracting this from $x^2y^2 = 729x + 729y$ gives $729x = 0$, or $x = 0$. Similar argument starting from $0 = 8x - 5832/y^2$ gives $y = 0$, and so the latter condition implies $x = y$ anyway.
Thus we can plug in $x = y$ into the equation

$$0 = 8y - 5832/x^2 = 8x - 5832/x^2,$$

which is to say $x^3 = 729$, or $x = 9$. This gives $y = 9$ immediately as well. This is a local (and global!) min, by checking the second derivative test. It is also a bummer because the requirement that $x \leq 7$. Thus we have to check the boundaries of our region.
Our region is $R = (0, 7] \times (0, \infty)$, which is neither closed nor bounded. We didn't really look at this in class so much, but we can consider it here. Since there is no critical point in R, we only have to check its boundaries. It should be clear that as x or y goes to 0 that $f(x, y) \to \infty$. Similarly as $y \to \infty$, f explodes. Since we are minimizing cost we do not need to consider these. Thus we look at the boundary $x = 7$. This gives $g(y) = f(7, y) = 56y + 5832(1/7 + 1/y)$, which we have to minimize. Use your 20A knowledge: $g'(y) = 56 - 5832/y^2$. Set this to zero to get the quadratic equation $0 = 7 - 729/y^2$, or $7y^2 = 729$, which has solution $y = \sqrt{729/7}$. By the second derivative test, this is a local min, so we have $\left(7, \sqrt{729/7}\right)$ as the local min.
This question was a little harder than we did in class, mostly because of the hand-wavey parts where the boundary is not closed and bounded. If you like, you could try the question where I explicitly mention in the problem formulation that we must take $(x, y) \in [1, 7] \times [1, 1000]$. Then you could do this exactly how we did it in class, but with more work than we did here because you would have to consider all four boundary edges separately, instead of throwing three of them away as we did here with the argument that $f \to \infty$.

2. Let $z = 117 - x - y$ and let $f(x, y) = x^a y^b (117 - x - y)^c$. Find the critical points:

$$\begin{aligned}
&\nabla f = \text{(work omitted)} \cdots \\
&= x^{a-1} y^{b-1} (117 - x - y)^{c-1} \langle ay(117 - x - y) - cxy, bx(117 - x - y) - cxy \rangle
\end{aligned}$$

Setting this to $\mathbf{0}$, and assuming $x, y, 117 - x - y$ are not zero, gives two equations:

$$ay(117 - x - y) = cxy, \quad \text{and} \quad bx(117 - x - y) = cxy$$

By setting these two equal to each other and cancelling, we get $ay = bx$, or $y = (b/a)x$. We also get, from the first equation, cancelling y, $cx = a(117 - x - y) = 117a - ax - bx$, and thus $x = 117a/(a + b + c)$, and $y = 117b/(a + b + c)$, which gives $z = 117 - x - y = 117c/(a + b + c)$.

Alternatively you can use the method of Lagrange Multipliers. Letting $g(x, y, z) = x + y + z$, you must maximize $f(x, y, z) = x^a y^b z^c$ subject to $g(x, y, z) = 117$. This gives four equations:

$$\begin{aligned}
\nabla f(x, y, z) &= \lambda \nabla g(x, y, z) \\
g(x, y, z) &= 117
\end{aligned}$$

(The first line above is three equations, one in each of x, y, z components.) This gives us

$$\begin{aligned}
a x^{a-1} y^b z^c &= \lambda \\
b x^a y^{b-1} z^c &= \lambda \\
c x^a y^b z^{c-1} &= \lambda \\
x + y + z &= 117
\end{aligned}$$

We can assume x, y, z are nonzero as if one of x, y, z are zero we have $f(x, y, z) = 0$, which does not maximize f. Now set the above equations for λ equal to each other and do some solving:

$$\begin{aligned}
a x^{a-1} y^b z^c &= \lambda = b x^a y^{b-1} z^c \\
a x^{a-1} y^b z^c &= b x^a y^{b-1} z^c \\
ay &= bx \\
y &= (b/a)x
\end{aligned}$$

Similarly $z = (c/a)x$. Plug this into the last equation to get

$$117 = x + y + z = x + (b/a)x + (c/a)x = \frac{a + b + c}{a} x$$

This solves for x, which then solves for y, z.

3. Use differentials. First rewrite

$$R_{total} = \left(\frac{1}{R_1} + \frac{1}{R_2} \right)^{-1} = \frac{R_1 R_2}{R_2 + R_1}$$

Now get the differential

$$\mathrm{d}R_{total} = \left(\frac{R_2}{R_1+R_2}\right)^2 \mathrm{d}R_1 + \left(\frac{R_1}{R_1+R_2}\right)^2 \mathrm{d}R_2$$

Now approximate

$$\begin{aligned}\Delta R_{total} \approx \mathrm{d}R_{total} &\approx \left(\frac{R_2}{R_1+R_2}\right)^2 \Delta R_1 + \left(\frac{R_1}{R_1+R_2}\right)^2 \Delta R_2 \\ &= \left(\frac{3000}{3500}\right)^2 25 + \left(\frac{500}{3500}\right)^2 100 \\ &= \frac{1000}{49}\Omega\end{aligned}$$

33. (§15.2) Given a function $f(x,y)$ and a rectangle R, evaluate

$$\iint\limits_R f(x,y)\,\mathrm{d}A$$

1. $f(x,y) = x+y, R = [0,1]\times[2,3]$.
2. $f(x,y) = x\sin y, R = [0,1]\times[0,\pi]$.
3. $f(x,y) = x/y, R = [0,1]\times[1,2]$.
4. $f(x,y) = \cos(x+2y), R = [0,\pi]\times[0,\pi/2]$.

Answer: 1. $\int_0^1\int_2^3 x+y \;\mathrm{d}y\,\mathrm{d}x = 3$.

2. $\int_0^1\int_0^\pi x\sin y \;\mathrm{d}y\,\mathrm{d}x = 1$.
3. $\int_0^1\int_1^2 x/y \;\mathrm{d}y\,\mathrm{d}x = \frac{1}{2}\ln 2$.
4. $\int_0^\pi\int_0^{\pi/2} \cos(x+2y) \;\mathrm{d}y\,\mathrm{d}x = -2$.

34. (§15.3) Given a function $f(x,y)$ and a general region in the plane, D, evaluate

$$\iint\limits_D f(x,y)\,\mathrm{d}A$$

1. $f(x,y) = x-y$, D is the triangle with corners $(0,0)$, $(0,5)$, $(3,2)$.
2. $f(x,y) = x+y, D = \left\{(x,y)\,\middle|\, x^2+y^2 \le 4\right\}$. (actually this is easy by symmetry considerations.) (and is better done with polar coordinates anyway.)
3. $f(x,y) = \sqrt{1-x^2}$, D is the triangle with corners $(0,0)$, $(1,0)$, $(1,1)$.
4. $f(x,y) = (x-y)^2$, D is the region between the curves $y=x^2$ and $y=x$.

Answer: 1. $\int_0^3\int_{\frac{2x}{3}}^{5-x} x-y \;\mathrm{d}y\,\mathrm{d}x = -10$.

2. $\int_0^2\int_0^{2\pi} (r\cos(\theta)+r\sin(\theta))\; r\,\mathrm{d}\theta\,\mathrm{d}r = 0$.
3. $\int_0^1\int_0^x \sqrt{1-x^2} \;\mathrm{d}y\,\mathrm{d}x = \int_0^1 x\sqrt{1-x^2}\,\mathrm{d}x = \frac{1}{3}$.
4. $\int_0^1\int_{x^2}^x (x-y)^2 \;\mathrm{d}y\,\mathrm{d}x = \int_0^1 x\sqrt{1-x^2}\,\mathrm{d}x = \frac{1}{420}$?

35. (§15.4) Given a function $f(x, y)$ and a general region in the plane, D, interpret the integral

$$\iint\limits_D f(x, y)\,\mathrm{d}A$$

as a polar integral and solve. *Note:* you should be able to recognize when it is appropriate or beneficial to evaluate a double integral in polar coordinates. *e.g.*,:

1. $f(x, y) = e^{x^2+y^2}, D = \left\{(x, y) \,\middle|\, x^2 + y^2 \leq 9\right\}$.
2. $f(x, y) = 2, D = \left\{(x, y) \,\middle|\, x^2 + y^2 \leq 1,\ y \geq x\right\}$.
3. $f(x, y) = x + y, D = \left\{(x, y) \,\middle|\, y \geq 0,\ 4 \leq x^2 + y^2 \leq 9\right\}$.
4. $f(x, y) = \sqrt{x^2 + y^2}, D = \{(r, \theta) \mid 0 \leq \theta \leq \pi/4,\ 0 \leq r \leq \cos 2\theta\}$.

Answer: 1. $\int_0^3 \int_0^{2\pi} e^{r^2}\, r\,\mathrm{d}\theta\,\mathrm{d}r = \pi\left(e^9 - 1\right)$.

2. $\int_0^1 \int_{\pi/4}^{5\pi/4} 2\,r\,\mathrm{d}\theta\,\mathrm{d}r = \pi$.
3. $\int_2^3 \int_0^\pi r\,(\cos\theta + \sin\theta)\; r\,\mathrm{d}\theta\,\mathrm{d}r = \frac{46}{3}$.
4. $\int_0^{\pi/4} \int_0^{\cos 2\theta} r\,r\,\mathrm{d}r\,\mathrm{d}\theta = \frac{1}{9}$.

36. (§15.5) Given a density function $\rho(x, y)$ for a lamina in the plane, D, find the center of mass of the lamina. *e.g.*,:

1. $\rho(x, y) = x + y + 2, D = \left\{(x, y) \,\middle|\, x^2 + y^2 \leq 1\right\}$.
2. $\rho(x, y) = 2, D = \left\{(x, y) \,\middle|\, 0 \leq x \leq 1,\ 0 \leq y \leq x^2\right\}$.
3. $\rho(x, y) = \frac{1}{\sqrt{x^2+y^2}}, D = \left\{(x, y) \,\middle|\, 0 \leq x,\ 1 \leq x^2 + y^2 \leq 4\right\}$. (*Hint:* try converting to polar integrals?)

Answer: 1. First we find the mass: $\iint_D \rho(x, y)\,\mathrm{d}A = 2\pi$. Then $\bar{x} = \frac{1}{m}\iint_D x\rho(x, y)\,\mathrm{d}A = \frac{1}{8}$. Similarly $\bar{y} = \frac{1}{m}\iint_D y\rho(x, y)\,\mathrm{d}A = \frac{1}{8}$. So the center of mass is $\left(\frac{1}{8}, \frac{1}{8}\right)$.

2. First we find the mass: $\iint_D \rho(x, y)\,\mathrm{d}A = \frac{2}{3}$. Then $\bar{x} = \frac{1}{m}\iint_D x\rho(x, y)\,\mathrm{d}A = \frac{3}{4}$. Similarly $\bar{y} = \frac{1}{m}\iint_D y\rho(x, y)\,\mathrm{d}A = \frac{3}{10}$. So the center of mass is $\left(\frac{3}{4}, \frac{3}{10}\right)$.
3. First we find the mass: $\iint_D \rho(x, y)\,\mathrm{d}A = \int_1^2 \int_{-\pi/2}^{\pi/2} \frac{1}{r}\,r\,\mathrm{d}\theta\,\mathrm{d}r = \pi$. $\bar{x} = \frac{1}{m}\iint_D x\rho(x, y)\,\mathrm{d}A = \frac{3}{\pi}$. Similarly $\bar{y} = \frac{1}{m}\iint_D y\rho(x, y)\,\mathrm{d}A = 0$. So the center of mass is $\left(\frac{3}{\pi}, 0\right)$.

37. (§15.7) Given a function $f(x, y, z)$, and some general region B in three space, evaluate

$$\iiint\limits_B f(x, y, z)\,\mathrm{d}V$$

1. $f(x, y, z) = 1 + xy,\ B = \{(x, y, z) \mid 0 \leq x \leq 1,\ 0 \leq y \leq 2,\ 1 \leq z \leq 2\}$.
2. $f(x, y, z) = yz\cos\left(x^5\right),\ B = \{(x, y, z) \mid 0 \leq x \leq 1,\ 0 \leq y \leq x,\ x \leq z \leq 2x\}$. (This is §15.7 #8)
3. $f(x, y, z) = xz,\ B = \{(x, y, z) \mid 0 \leq x \leq 1,\ 0 \leq y \leq x,\ 0 \leq z \leq 6 - x - y\}$.

Answer: 1. $\int_0^1 \int_0^2 \int_1^2 1 + xy\ \mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \int_0^2 1 + xy\ \mathrm{d}y\,\mathrm{d}x = \int_0^1 2 + 2x\ \mathrm{d}x = 3$.

2. $\int_0^1 \int_0^x \int_x^{2x} yz\cos x^5\ \mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \int_0^x \frac{3}{2}x^2 y\cos x^5\ \mathrm{d}y\,\mathrm{d}x = \int_0^1 \frac{3}{4}x^4\cos x^5\ \mathrm{d}x = \frac{3}{20}\sin 1$.

3.

$$\int_0^1\int_0^x\int_0^{6-x-y} xz\;\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x = \int_0^1\int_0^x \frac{x}{2}\,(6-x-y)^2\;\;\mathrm{d}y\,\mathrm{d}x$$

$$= \int_0^1 \frac{1}{6}\left(36x^2 3 - 18x^3 + 7x^4\right)\;\mathrm{d}x = \frac{89}{60}$$

38. (§12.7) Convert the following points, given in cylindrical coordinates, to cartesian coordinates, and to spherical coordinates:

$$P = (1, \pi/3, 1) \qquad Q = (2, \pi, 0) \qquad R = \left(1, \pi/6, -\sqrt{3}\right)$$

Answer:

$$P = \left(1/2, \sqrt{3}/2, 1\right) \qquad Q = (-2, 0, 0) \qquad R = \left(\sqrt{3}/2, 1/2, -\sqrt{3}\right)$$

39. (§12.7) Convert the following points, given in spherical coordinates, to cartesian coordinates, and to cylindrical coordinates:

$$P = [(\rho, \theta, \phi) = (1, \pi/2, \pi/3)] \qquad Q = [(\rho, \theta, \phi) = (2, 3\pi/2, \pi/2)]$$

Answer:

$$P = \left(0, \sqrt{3}/2, 1/2\right) \qquad Q = (0, -2, 0)$$

40. (§15.8) Rewrite the triple integral

$$\iiint_R f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

in cylindrical coordinates. (**Note:** the resulting integrals might be very difficult or impossible.)

1. $f(x, y, z) = 3x^2+3y^2+7z$, and R is the region with $z \le 5$, $x^2+y^2 \le 4$, $x^2+y^2+z^2 \ge 4$.
2. $f(x, y, z) = 1$, and R is the region with $z \ge \sqrt{x^2+y^2}, x^2+y^2+z^2 \le 32$. (Note: since you are integrating 1, the integral gives the volume of R.)
3. $f(x, y, z) = ze^{x^2+y^2}$, R is the region with $x \ge 0$, $y \ge 0$, $z \ge 0, z \le 9 - x^2 - y^2$.

Answer: 1. $\int_0^{2\pi}\int_0^2\int_{4-r^2}^5 \left(3r^2 + 7z\right) r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$

2. $\int_0^{2\pi}\int_0^4\int_r^{\sqrt{32-r^2}} 1r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$
3. $\int_0^{\pi/2}\int_0^3\int_0^{9-r^2} ze^{r^2} r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$

41. (§15.8) Rewrite the triple integral

$$\iiint_R f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

in spherical coordinates. (Again the resulting integrals might be difficult.)

1. $f(x,y,z) = 13\left(x^2+y^2+z^2\right)^{3/2}$, and R is the region $x^2+y^2+z^2 \le 9$.
2. $f(x,y,z) = 5\left(x^2+y^2\right)$, and R is the region $x \ge 0, y \ge 0, z \ge 0, 4 \le x^2+y^2+z^2 \le 16$.

Answer: 1. $\int_0^{2\pi}\int_0^{\pi}\int_0^{3} 13\left(\rho^2\right)^{3/2} \rho^2 \sin\phi \, \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}\theta$

2. $\int_0^{\pi/2}\int_0^{\pi/2}\int_2^{4} 5\rho^2 \sin^2(\phi)\rho^2 \sin\phi \, \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}\theta$

42. (misc) On the exam you may not be told what technique is appropriate for a given problem: you must decide this for yourself. In this category are the following miscellaneous problems:

1. Find the volume of the tetrahedron with corners $(0,0,0)$, $(2,0,0)$, $(0,1,0)$, $(0,0,3)$.
2. Find numbers a, b, c with $a+b+c = 100$, such that a^2bc is a maximum.
3. Let $p(x,y,z) = x^2y + z^2x$ represent the density of plankton in the ocean. A whale is located at the point $(1,2,0)$. In which direction should the whale move to maximize the increase in plankton concentration?
4. Maximize $f(x,y) = 2x^3 + y^3$ subject to $x^2+y^2 = 4$.

Answer: 1. This is $\int_0^2\int_0^{1-\frac{x}{2}}\int_0^{\frac{6-3x-6y}{2}} 1 \, \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}x = \int_0^2\int_0^{1-\frac{x}{2}} \frac{6-3x-6y}{2} \, \mathrm{d}y \, \mathrm{d}x =$ $\int_0^2 \frac{3}{2} - \frac{3x}{2} + \frac{3x^2}{8} \, \mathrm{d}x = 1.$

2. Maximize the function $f(a,b) = a^2b(100-a-b)$. Find that it has a maximum at $a = 50, b = 25$. Then $c = 25$.
3. The gradient is the direction of maximal increase. In this case we have $\nabla p(x,y,z) = \left\langle 2xy + z^2, x^2, 2zx \right\rangle$. At the point $(1,2,0)$ this takes value $\langle 4,1,0\rangle$. To denote a direction, we unitize this vector to get $\frac{1}{\sqrt{17}}\langle 4,1,0\rangle$.
4. While this region is simple enough for us to tackle this problem with a different method, in general we will have to use the method of Lagrange Multipliers. We first let $g(x,y) = x^2+y^2$. We then look for x, y, λ such that $\nabla f(x,y) = \lambda \nabla g(x,y)$, and $g(x,y) = 4$. When we write the gradient equation as two equations, this gives three equations:

$$\begin{aligned} f_x(x,y) &= \lambda g_x(x,y) \\ f_y(x,y) &= \lambda g_y(x,y) \\ g(x,y) &= 4 \end{aligned}$$

In this particular case we have

$$\begin{aligned} 6x^2 &= 2\lambda x \\ 3y^2 &= 2\lambda y \\ x^2+y^2 &= 4. \end{aligned}$$

Sadly there is no general method for attacking the equations that Lagrange's Method gives us. What works in this case may not work in others. Consider the first equation, which we can write as $3x^2 = \lambda x$. This has solutions $x = 0$ and $3x = \lambda$.
If we consider the case $x = 0$, and plug into the third equation, $x^2+y^2 = 4$, we get $y^2 = 4$, so $y = \pm 2$. This gives points $(0,-2)$ and $(0,2)$.

For the moment ignore the other case for the first equation and consider the second equation, which, similarly, has solutions $y = 0$ and $y = 2\lambda/3$. If $y = 0$, then, as above, $x = \pm 2$, so we have points $(-2, 0)$ and $(2, 0)$.
Now return to the two ignored cases, assuming $x \neq 0$ and $y \neq 0$, thus $3x = \lambda$ and $y = 2\lambda/3$. Together these give $y = 2x$. Now plug this into the third equation to get

$$4 = x^2 + y^2 = x^2 + (2x)^2 = 5x^2,$$

so $x = \pm\sqrt{4/5}$, and $y = 2x$. This gives points $\left(-\sqrt{4/5}, -2\sqrt{4/5}\right)$ and $\left(\sqrt{4/5}, 2\sqrt{4/5}\right)$.
Now compute the table.

(x, y)	$f(x, y)$
$(0, -2)$	-8
$(0, 2)$	8
$(-2, 0)$	-16
$(2, 0)$	16
$\left(-\sqrt{4/5}, -2\sqrt{4/5}\right)$	$-8\sqrt{4/5}$
$\left(\sqrt{4/5}, 2\sqrt{4/5}\right)$	$8\sqrt{4/5}$

The maximal value is 16, and the minimal value is -16.