# 2005W M20E Exam 1 Preparation

The first midterm exam is Monday January 31, during the class period. *You must bring a blue book to the exam.* Blue books are available from the bookstore. You also need to bring your student ID card or other form of ID (driver's license, passport, etc.)

The following formula will be provided on your exam:

$$\nabla \quad \text{abbreviates} \quad \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}$$

You should be prepared to answer at least the following questions:

**1. (§1.1–1.3)** What's the difference between a scalar and a vector?

**2. (§1.1–1.3)** Let $\boldsymbol{u} = \langle 1, 2, -3 \rangle$, $\boldsymbol{v} = \langle -2, 4, 1 \rangle$, $\boldsymbol{w} = \langle -3, 0, 1 \rangle$.

1. Find $5\boldsymbol{u}$.
2. Find $\boldsymbol{u} \cdot \boldsymbol{u}$.
3. Find $\|\boldsymbol{u}\|$.
4. Find a unit length vector in the same direction as $\boldsymbol{u}$.
5. Find $\|\alpha\boldsymbol{u}\|$, where $\alpha$ is a real number.
6. Find $\boldsymbol{u} \cdot \boldsymbol{v}$.
7. Find the angle subtended by $\boldsymbol{u}$ and $\boldsymbol{v}$.
8. Find $\boldsymbol{v} \times \boldsymbol{w}$.
9. Find a vector perpendicular to both $\boldsymbol{v}$ and $\boldsymbol{w}$.
10. Find a unit length vector perpendicular to both $\boldsymbol{v}$ and $\boldsymbol{w}$.
11. Find $\boldsymbol{u} \times \boldsymbol{v}$.
12. Find $\boldsymbol{w} \cdot (\boldsymbol{u} \times \boldsymbol{v})$.
13. Find $\boldsymbol{u} \times \boldsymbol{u}$.

**3. (§1.1–1.3)** Let $\boldsymbol{u} = \langle -2, 1, 4 \rangle$, and let $P$ be the point $(0, 4, 1)$.

1. Find a path $\boldsymbol{l}(t)$ which traces out the line through $P$ and parallel to $\boldsymbol{u}$.
2. Find the equation of the plane containing $P$ with normal $\boldsymbol{u}$.
3. Find a path $\boldsymbol{m}(t)$ which traces out a line through $P$ and perpendicular to $\boldsymbol{u}$. (*Hint:* There are many answers. One way to do this is pick some arbitrary vector $\boldsymbol{v}$ not parallel to $\boldsymbol{u}$, then find a vector perpendicular to both $\boldsymbol{u}$ and $\boldsymbol{v}$.)

**4. (§1.4)** Convert the following points, given in cylindrical coordinates, to cartesian coordinates, and to spherical coordinates:

$$P = (1, \pi/3, 1) \qquad Q = (2, \pi, 0) \qquad R = \left(1, \pi/6, -\sqrt{3}\right)$$

**5. (§1.4)** Convert the following points, given in spherical coordinates, to cartesian coordinates, and to cylindrical coordinates:

$$P = [(\rho, \theta, \phi) = (1, \pi/2, \pi/3)] \qquad Q = [(\rho, \theta, \phi) = (2, 3\pi/2, \pi/2)]$$

**6. (§2.1)** Define a level set for a function $f\colon \mathbb{R}^n \to \mathbb{R}$

**7. (§2.1)** Describe the level sets of the following functions. Where appropriate, graph a few of the level sets.

1. $f(x, y) = x^2 + y^2$.
2. $g(x, y) = 2x + y$.
3. $h(x, y) = xy$.
4. $l(x, y, z) = x - y + 2z$.
5. $m(x, y, z) = x^2 + y^2$.
6. $n(x, y, z) = x^2 + y^2 + z^2$.
7. $p(x, y, z) = x^2 + y^2 - z$.

**8. (§2.1)** Sketch the graphs of the following functions:

1. $f(x, y) = x^2 + y^2$.
2. $g(x, y) = 2x + y$.

**9. (§2.3)** Find $\nabla f$ for

1. $f(x, y) = x^2 + y^2$.
2. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$.
3. $f(x, y) = e^{x+y}$.
4. $f(x, y, z) = e^{x+y} \cos z + y^2$.

**10. (§2.3)** Find the equation of the tangent plane to the graph of $f$ at the point $P$ for

1. $f(x, y) = x^2 + y^2$, $P = (3, 4, 25)$.
2. $f(x, y) = e^{x+y}$, $P = \big(2, 1, e^3\big)$.

**11. (§2.3)** In your own words, what does it mean for a function $f\colon \mathbb{R}^n \to \mathbb{R}$ to be "differentiable" at a point $x_0$? If $f$ is differentiable at $x_0$, does $f$ have to be smooth at this point? Is there a unique tangent plane at this point?

**12. (§2.3)** Remember the theorem: if all the partial derivatives of $f\colon \mathbb{R}^n \to \mathbb{R}$ exist and are continuous at $x_0$, then $f$ is differentiable at $x_0$.

**13. (§2.4)** What kind of curve does $\boldsymbol{c}(t)$ trace out for

1. $\boldsymbol{c}(t) = \langle 1 + 2t, 3, 4 - t\rangle$
2. $\boldsymbol{c}(t) = \langle \cos 2t, \sin 2t\rangle$.
3. $\boldsymbol{c}(t) = \langle \cos t, \sin t, 3t\rangle$.

**14. (§2.4)** Given the path $\boldsymbol{c}(t)$, find its velocity function $\boldsymbol{c}'(t)$, and its speed function $\|\boldsymbol{c}'(t)\|$

1. $\boldsymbol{c}(t) = \langle 1 + 3t, 3, 1 - 4t\rangle$
2. $\boldsymbol{c}(t) = \langle \cos t, \sin t, 3t\rangle$.
3. $\boldsymbol{c}(t) = \big\langle t^2, t^3, t\big\rangle$.
4. $\boldsymbol{c}(t) = \big\langle t, t^3 + 1/(4t)\big\rangle$.
5. $\boldsymbol{c}(t) = \big\langle t/\sqrt{2}, t/\sqrt{2}, \ln(\sec x)\big\rangle$.

**15. (§2.4)** Given the path $\boldsymbol{c}(t)$, and a time $t_0$, find the tangent vector to $\boldsymbol{c}(t_0)$ for

1. $\boldsymbol{c}(t) = \langle 1 + 3t, 3, 1 - 4t\rangle$, $t_0 = 1$.
2. $\boldsymbol{c}(t) = \big\langle t^2, -3t, 1 + t^3\big\rangle$, $t_0 = 2$.

**16. (§2.5)** Given the path $\boldsymbol{c}(t)$ in $\mathbb{R}^n$ and a real valued function $f(\boldsymbol{x})$ on n variables, find $\frac{\mathrm{d}}{\mathrm{d}t}\left[f \circ \boldsymbol{c}(t)\right]$

1. $\boldsymbol{c}(t) = \langle 1+3t, 3, 1-4t \rangle$ , $f(x,y,z) = xy - yz$.
2. $\boldsymbol{c}(t) = \langle \cos t, \sin t, t \rangle$ , $f(x,y,z) = x^2 + y^2 - z$
3. $\boldsymbol{c}(t) = \left\langle t, t^2, t^3 \right\rangle$ , $f(x,y,z) = xyz + e^{xy}$.

**17. (§2.6)** Given the real valued function $f(\boldsymbol{x})\colon \mathbb{R}^n \to \mathbb{R}$, and the vector $\boldsymbol{v}$, find the directional derivative of $f$ along $\boldsymbol{v}$ at a given point, for

1. $f(x,y) = x^2 + y^2$, $\boldsymbol{v} = \langle 1, 3 \rangle$ , at the point $(3,4)$.
2. $f(x,y,z) = \sqrt{x^2 + y^2 + z^2}$, $\boldsymbol{v} = \langle -1, 0.5, 3 \rangle$, at the point $(1,0,-1)$.
3. $f(x,y) = e^{x+y}$, $\boldsymbol{v} = \left\langle \sqrt{2}/2, \sqrt{2}/2 \right\rangle$, at the point $(1,-1)$.
4. $f(x,y,z) = e^{x+y}\cos z + y^2$, $\boldsymbol{v} = \left\langle 1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3} \right\rangle$, at the point $(2,1,\pi/2)$.
5. $f(x,y,z) = x + y \ln z$, $\boldsymbol{v} = \langle 1/3, 1, -1 \rangle$ at the point $(1,2,0)$.

**18. (§2.6)** Given the real valued function $f(\boldsymbol{x})\colon \mathbb{R}^n \to \mathbb{R}$, and a point in $\mathbb{R}^n$, find the direction (*i.e.*, unit vector) of maximal increase of $f$ at that point for

1. $f(x,y) = x^2 + y^2$, at the point $(3,4)$.
2. $f(x,y) = xy + 1/(xy)$, at the point $(1,2)$.
3. $f(x,y,z) = xyz$, at the point $(1,0,2)$.
4. $f(x,y,z) = 2x + 3y - 4z$, at the point $(1,-1,5)$.
5. $f(x,y,z) = x^2 + y^2 - z$, at the point $(0,1,1)$.

**19. (§3.2)** Find the linear Taylor formula (*i.e.*, linearization) of $f(x,y) = x^2 + y^2$ at the point $(1,2)$.

**20. (§3.2)** Find the quadratic Taylor formula of $f(x,y) = x^2 + y^2$ at the point $(1,2)$.

**21. (§4.2)** Given path $\boldsymbol{c}(t)$, find the arc length of the path for $t$ between $t_0$ and $t_1$, for

1. $\boldsymbol{c}(t) = \langle 1+3t, 3, 1-4t \rangle$, $t_0 = 0$, $t_1 = 3$.
2. $\boldsymbol{c}(t) = \langle \cos t, \sin t, 3t \rangle$, $t_0 = 0$, $t_1 = 4\pi$.
3. $\boldsymbol{c}(t) = \left\langle t/\sqrt{2}, t/\sqrt{2}, \ln(\sec x) \right\rangle$, $t_0 = 0$, $t_1 = \pi/3$.
4. $\boldsymbol{c}(t) = \left\langle t, t^3/3 + 1/(4t) \right\rangle$, $t_0 = 1$, $t_1 = 2$.

**22. (§4.4)** Match four of the following vector fields to their graphical representation in Figure 1. (Two of the following fields are not plotted.)

1. $\boldsymbol{F}(x,y) = \langle y, x \rangle$.
2. $\boldsymbol{F}(x,y) = \left\langle x^2, y^2 \right\rangle$.
3. $\boldsymbol{F}(x,y) = \langle 2, 5 \rangle$.
4. $\boldsymbol{F}(x,y) = \langle y, -x \rangle$.
5. $\boldsymbol{F}(x,y) = \langle 2x, 2y \rangle$.
6. $\boldsymbol{F}(x,y) = \langle -y, x \rangle$.

**23. (§4.4)** Draw some flow lines for the vector fields in Figure 1.

**24. (§4.4)** Which of the vector fields plotted in Figure 1 might be gradient fields, that is of the form $\nabla\phi$ for some scalar field $\phi$? For those that are, try to draw some level sets for the function $\phi$ over the vector field.

**25. (§4.4)** Given the vector field $\boldsymbol{F}(x,y,z)$, find div $\boldsymbol{F} = \nabla \cdot \boldsymbol{F}$, and **curl** $\boldsymbol{F} = \nabla \times \boldsymbol{F}$ for

1. $\boldsymbol{F}(x,y,z) = \left\langle xy + z, x^2 - y, z^2 - 3 \right\rangle$.
2. $\boldsymbol{F}(x,y,z) = \langle yz, xz, xy \rangle$.
3. $\boldsymbol{F}(x,y,z) = \langle 2x, 2y, -1 \rangle$.
4. $\boldsymbol{F}(x,y,z) = \langle y, -x, 0 \rangle$.
5. $\boldsymbol{F} = \nabla \phi$, where $\phi(x,y,z) = 2xy^2 - ze^y$.
6. $\boldsymbol{F} = \nabla \times \boldsymbol{G}$, where $\boldsymbol{G}(x,y,z) = \left\langle xy + z, x^2 - y, z^2 - 3 \right\rangle$.

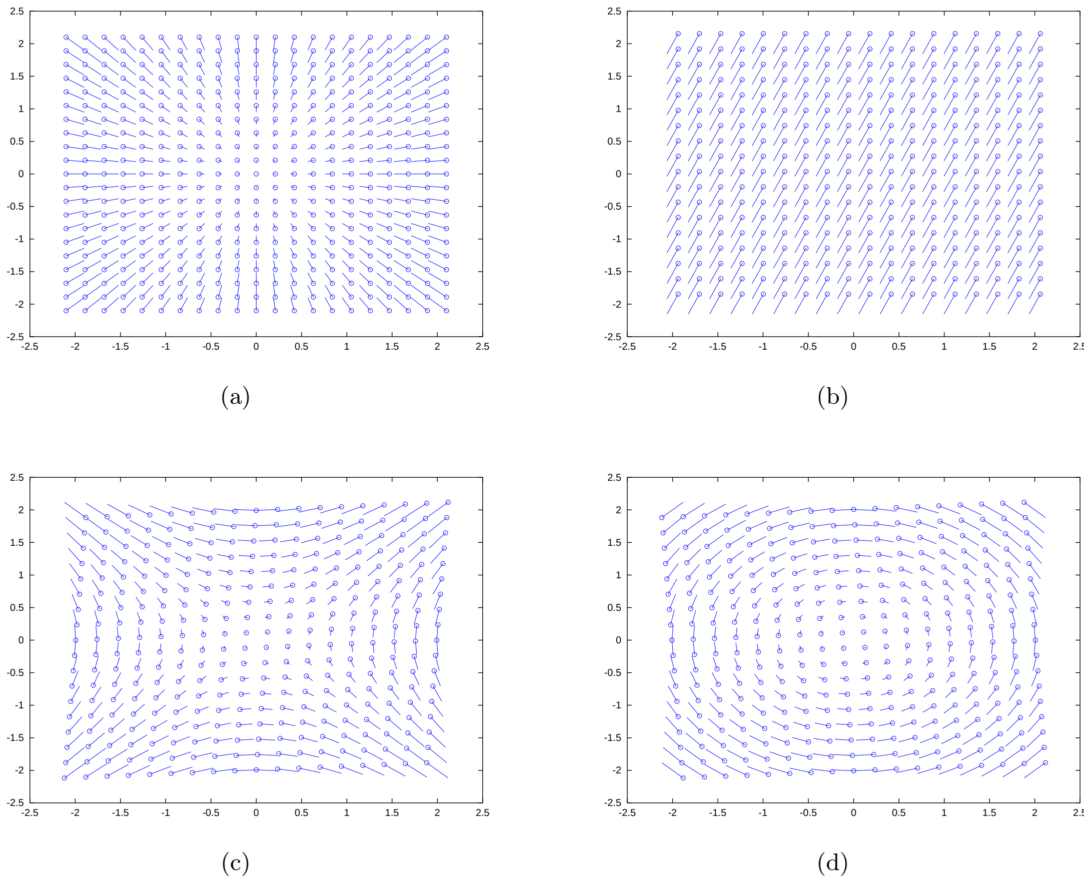


Figure 1: Four vector fields. Circles represent the tips of the vectors (proper arrowheads were not available).