2005W M20E Exam 1 Preparation

The first midterm exam is Monday January 31, during the class period. *You must bring a blue book to the exam.* Blue books are available from the bookstore. You also need to bring your student ID card or other form of ID (driver's license, passport, etc.)

The following formula will be provided on your exam:

$$\nabla \quad \text{abbreviates} \quad \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}$$

You should be prepared to answer at least the following questions:

1. (§1.1–1.3) What's the difference between a scalar and a vector?

Answer: A vector comprises a direction and a scalar magnitude. A scalar can be conceived of as a vector in 1 dimension, and thus has trivial direction (*i.e.*, in the positive direction).

2. (§1.1–1.3) Let $\boldsymbol{u} = \langle 1, 2, -3 \rangle$, $\boldsymbol{v} = \langle -2, 4, 1 \rangle$, $\boldsymbol{w} = \langle -3, 0, 1 \rangle$.

1. Find $5\boldsymbol{u}$.
2. Find $\boldsymbol{u} \cdot \boldsymbol{u}$.
3. Find $\|\boldsymbol{u}\|$.
4. Find a unit length vector in the same direction as $\boldsymbol{u}$.
5. Find $\|\alpha \boldsymbol{u}\|$, where α is a real number.
6. Find $\boldsymbol{u} \cdot \boldsymbol{v}$.
7. Find the angle subtended by $\boldsymbol{u}$ and $\boldsymbol{v}$.
8. Find $\boldsymbol{v} \times \boldsymbol{w}$.
9. Find a vector perpendicular to both $\boldsymbol{v}$ and $\boldsymbol{w}$.
10. Find a unit length vector perpendicular to both $\boldsymbol{v}$ and $\boldsymbol{w}$.
11. Find $\boldsymbol{u} \times \boldsymbol{v}$.
12. Find $\boldsymbol{w} \cdot (\boldsymbol{u} \times \boldsymbol{v})$.
13. Find $\boldsymbol{u} \times \boldsymbol{u}$.

Answer:
1. $\langle 5, 10, -15 \rangle$
2. 14
3. $\sqrt{14}$
4. $(1/\sqrt{14})\boldsymbol{u} = \left\langle 1/\sqrt{14}, 2/\sqrt{14}, -3/\sqrt{14} \right\rangle$.
5. $|\alpha| \sqrt{14}$
6. $-2 + 8 - 3 = 3$
7. $\arccos\left(\boldsymbol{u} \cdot \boldsymbol{v} / \|\boldsymbol{u}\| \, \|\boldsymbol{v}\|\right) = \arccos\left(3/\sqrt{14}\sqrt{21}\right) \approx 80°$
8. $\boldsymbol{v} \times \boldsymbol{w} = \langle 4, -1, 12 \rangle$
9. $\boldsymbol{v} \times \boldsymbol{w}$ is perpendicular to both $\boldsymbol{v}$ and $\boldsymbol{w}$.
10. $\boldsymbol{v} \times \boldsymbol{w} / \|\boldsymbol{v} \times \boldsymbol{w}\| = \langle 4, -1, 12 \rangle / \sqrt{161} = \left\langle 4/\sqrt{161}, -1/\sqrt{161}, 12/\sqrt{161} \right\rangle$.

3. (§1.1–1.3) Let $\boldsymbol{u} = \langle -2, 1, 4 \rangle$, and let P be the point $(0, 4, 1)$.

1. Find a path $\boldsymbol{l}(t)$ which traces out the line through P and parallel to $\boldsymbol{u}$.
2. Find the equation of the plane containing P with normal $\boldsymbol{u}$.

3. Find a path $\boldsymbol{m}(t)$ which traces out a line through P and perpendicular to $\boldsymbol{u}$. (*Hint:* There are many answers. One way to do this is pick some arbitrary vector $\boldsymbol{v}$ not parallel to $\boldsymbol{u}$, then find a vector perpendicular to both $\boldsymbol{u}$ and $\boldsymbol{v}$.)

Answer:

1. One answer is $\boldsymbol{l}(t) = \langle 0, 4, 1 \rangle + t \langle -2, 1, 4 \rangle = \langle -2t, 4 + t, 1 + 4t \rangle$.
2. Planes are of the form $ax + by + cz + d = 0$, where $\langle a, b, c \rangle$ (and any scalar multiple of it) are normal to the plane. So the answer should be of the form $-2x + y + 4z + d = 0$. Plug in P as (x, y, z) to find the value of d. You should get $-2x + y + 4z - 8 = 0$.
3. First I will let $\boldsymbol{v} = \langle 0, 0, 1 \rangle$, just for simplicity. Then I compute $\boldsymbol{w} = \boldsymbol{u} \times \boldsymbol{v} = \langle 1, 2, 0 \rangle$, which is perpendicular to both $\boldsymbol{u}$ and $\boldsymbol{v}$. I only care that it is perpendicular to $\boldsymbol{u}$. If I had picked the "wrong" $\boldsymbol{v}$, this cross product would have been $\mathbf{0}$, and I would have known something was wrong and picked a different $\boldsymbol{v}$. Then I do as in the first part, getting $\boldsymbol{m}(t) = \langle 0, 4, 1 \rangle + t \langle 1, 2, 0 \rangle$.

4. (§1.4) Convert the following points, given in cylindrical coordinates, to cartesian coordinates, and to spherical coordinates:

$$P = (1, \pi/3, 1) \qquad Q = (2, \pi, 0) \qquad R = \left(1, \pi/6, -\sqrt{3}\right)$$

Answer:

$$P = \left(1/2, \sqrt{3}/2, 1\right) \qquad Q = (-2, 0, 0) \qquad R = \left(\sqrt{3}/2, 1/2, -\sqrt{3}\right)$$

5. (§1.4) Convert the following points, given in spherical coordinates, to cartesian coordinates, and to cylindrical coordinates:

$$P = [(\rho, \theta, \phi) = (1, \pi/2, \pi/3)] \qquad Q = [(\rho, \theta, \phi) = (2, 3\pi/2, \pi/2)]$$

Answer:

$$P = \left(0, \sqrt{3}/2, 1/2\right) \qquad Q = (0, -2, 0)$$

6. (§2.1) Define a level set for a function $f \colon \mathbb{R}^n \to \mathbb{R}$

Answer: A level set is a set of the form $\{\boldsymbol{x} \in \mathbb{R}^n \mid f(\boldsymbol{x}) = k\}$, that is, all points in $\mathbb{R}^n$ where f takes the value k. There is a different level set for each real constant k.

7. (§2.1) Describe the level sets of the following functions. Where appropriate, graph a few of the level sets.

1. $f(x, y) = x^2 + y^2$.
2. $g(x, y) = 2x + y$.
3. $h(x, y) = xy$.
4. $l(x, y, z) = x - y + 2z$.
5. $m(x, y, z) = x^2 + y^2$.
6. $n(x, y, z) = x^2 + y^2 + z^2$.
7. $p(x, y, z) = x^2 + y^2 - z$.

Answer:

1. $f(x, y) = x^2 + y^2$: concentric circles centered at the origin.

2. $g(x, y) = 2x + y$: lines with slope -2.
3. $h(x, y) = xy$: these should look like rotated hyperbolæ, or the graphs of $y = k/x$ for different constants k.
4. $l(x, y, z) = x - y + 2z$: planes with normal $\langle 1, -1, 2\rangle$.
5. $m(x, y, z) = x^2 + y^2$: cylinders with the z-axis as the cylindrical axis.
6. $n(x, y, z) = x^2 + y^2 + z^2$: spheres centered at the origin.
7. $p(x, y, z) = x^2 + y^2 - z$: paraboloids opening in the positive z-direction, and shifted above or below the xy plane.

8. (§2.1) Sketch the graphs of the following functions:

1. $f(x, y) = x^2 + y^2$.
2. $g(x, y) = 2x + y$.

Answer: 1. $f(x, y) = x^2 + y^2$. This is a paraboloid.

2. $g(x, y) = 2x + y$. This is a plane with normal $\langle 2, 1, -1\rangle$, and going through the origin.

9. (§2.3) Find ∇f for

1. $f(x, y) = x^2 + y^2$.
2. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$.
3. $f(x, y) = e^{x+y}$.
4. $f(x, y, z) = e^{x+y} \cos z + y^2$.

Answer: 1. $\nabla f(x, y) = \langle 2x, 2y\rangle$.

2. $\nabla f(x, y, z) = \left\langle \frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}} \right\rangle = \langle x, y, z\rangle / f(x, y, z)$.
3. $\nabla f(x, y) = \langle e^{x+y}, e^{x+y}\rangle = \langle f(x, y), f(x, y)\rangle$.
4. $\nabla f(x, y, z) = \langle e^{x+y} \cos z, e^{x+y} \cos z + 2y, -e^{x+y} \sin z\rangle$.

10. (§2.3) Find the equation of the tangent plane to the graph of f at the point P for

1. $f(x, y) = x^2 + y^2$, $P = (3, 4, 25)$.
2. $f(x, y) = e^{x+y}$, $P = \left(2, 1, e^3\right)$.

Answer: If $P = (x_0, y_0, f(x_0, y_0))$, the general form is

$$z - f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \langle x - x_0, y - y_0\rangle$$

1. The plane is $z - 25 = \langle 6, 8\rangle \cdot \langle x - 3, y - 4\rangle$. This can be simplified to the usual $ax + by + cz + d = 0$ form.
2. The plane is $z - e^3 = \left\langle e^3, e^3\right\rangle \cdot \langle x - 1, y - 1\rangle$.

11. (§2.3) In your own words, what does it mean for a function $f\colon \mathbb{R}^n \to \mathbb{R}$ to be "differentiable" at a point x_0? If f is differentiable at x_0, does f have to be smooth at this point? Is there a unique tangent plane at this point?

Answer: The function is differentiable if it is smooth around x_0. If all the parial derivatives of f exist and are continuous at x_0, then it is differentiable. Another characterization is that the difference between the function and its tangent (hyper)plane at a point x is, in the limit, "smaller" than the distance from x to x_0. Differentiability gives a certain amount of smoothness. Moreover, if f is differentiable at a point it does have a unique tangent plane. Your words may vary.

12. (§2.3) Remember the theorem: if all the partial derivatives of $f\colon \mathbb{R}^n \to \mathbb{R}$ exist and are continuous at x_0, then f is differentiable at x_0.

Answer: Noted.

13. (§2.4) What kind of curve does $\boldsymbol{c}(t)$ trace out for

1. $\boldsymbol{c}(t) = \langle 1+2t, 3, 4-t\rangle$
2. $\boldsymbol{c}(t) = \langle \cos 2t, \sin 2t\rangle$.
3. $\boldsymbol{c}(t) = \langle \cos t, \sin t, 3t\rangle$.

Answer:
1. $\boldsymbol{c}(t) = \langle 1+2t, 3, 4-t\rangle$: a straight line parallel to $\langle 2, 0, -1\rangle$.
2. $\boldsymbol{c}(t) = \langle \cos 2t, \sin 2t\rangle$: a circle in the plane centered at the origin.
3. $\boldsymbol{c}(t) = \langle \cos t, \sin t, 3t\rangle$: a helix.

14. (§2.4) Given the path $\boldsymbol{c}(t)$, find its velocity function $\boldsymbol{c}'(t)$, and its speed function $\|\boldsymbol{c}'(t)\|$

1. $\boldsymbol{c}(t) = \langle 1+3t, 3, 1-4t\rangle$
2. $\boldsymbol{c}(t) = \langle \cos t, \sin t, 3t\rangle$.
3. $\boldsymbol{c}(t) = \left\langle t^2, t^3, t\right\rangle$.
4. $\boldsymbol{c}(t) = \left\langle t, t^3 + 1/(4t)\right\rangle$.
5. $\boldsymbol{c}(t) = \left\langle t/\sqrt{2}, t/\sqrt{2}, \ln(\sec t)\right\rangle$.

Answer:
1. $\boldsymbol{c}'(t) = \langle 3, 0, -4\rangle$, and $\|\boldsymbol{c}'(t)\| = 5$.
2. $\boldsymbol{c}'(t) = \langle -\sin t, \cos t, 3\rangle$, and $\|\boldsymbol{c}'(t)\| = \sqrt{10}$.
3. $\boldsymbol{c}'(t) = \left\langle 2t, 3t^2, 1\right\rangle$, and $\|\boldsymbol{c}'(t)\| = \sqrt{4t^2 + 9t^4 + 1}$.
4. $\boldsymbol{c}'(t) = \left\langle 1, 3t^2 - 1/(4t^2)\right\rangle$, and $\|\boldsymbol{c}'(t)\| = \sqrt{1 + 9t^4 - 3/2 + 1/(16t^4)}$.
5. $\boldsymbol{c}'(t) = \left\langle 1/\sqrt{2}, 1/\sqrt{2}, \tan t\right\rangle$, and $\|\boldsymbol{c}'(t)\| = \sqrt{1+\tan^2 t} = |\sec t|$.

15. (§2.4) Given the path $\boldsymbol{c}(t)$, and a time t_0, find the tangent vector to $\boldsymbol{c}(t_0)$ for

1. $\boldsymbol{c}(t) = \langle 1+3t, 3, 1-4t\rangle$, $t_0 = 1$.
2. $\boldsymbol{c}(t) = \left\langle t^2, -3t, 1+t^3\right\rangle$, $t_0 = 2$.

Answer:
1. $\boldsymbol{c}(t) = \langle 3, 0, -4\rangle$.
2. $\boldsymbol{c}(t) = \langle 4, -3, 12\rangle$.

16. (§2.5) Given the path $\boldsymbol{c}(t)$ in $\mathbb{R}^n$ and a real valued function $f(\boldsymbol{x})$ on n variables, find $\frac{\mathrm{d}}{\mathrm{d}t}\left[f \circ \boldsymbol{c}(t)\right]$

1. $\boldsymbol{c}(t) = \langle 1+3t, 3, 1-4t\rangle$, $f(x,y,z) = xy - yz$.
2. $\boldsymbol{c}(t) = \langle \cos t, \sin t, t\rangle$, $f(x,y,z) = x^2 + y^2 - z$
3. $\boldsymbol{c}(t) = \left\langle t, t^2, t^3\right\rangle$, $f(x,y,z) = xyz + e^{xy}$.

Answer: You *could* do this by substitution and use the chain rule from M20A. However, in this class we should use the multivariable chain rule:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[f \circ \boldsymbol{c}(t)\right] = \nabla f(\boldsymbol{c}(t)) \cdot \boldsymbol{c}'(t)$$

The difficult thing is to remember to plug $\vec{c}(t)$ into f as the x, y and z.

1. We get $\nabla f(x,y,z) = \langle y, x-z, -y\rangle$, and $\boldsymbol{c}'(t) = \langle 3, 0, -4\rangle$. So
$$\frac{\mathrm{d}}{\mathrm{d}t}[f \circ \boldsymbol{c}(t)] = \langle 3, 1+3t-(1-4t), -3\rangle \cdot \langle 3, 0, -4\rangle = 9 + 12 = 21$$
2. We get $\nabla f(x,y,z) = \langle 2x, 2y, -1\rangle$, and $\boldsymbol{c}'(t) = \langle -\sin t, \cos t, 1\rangle$. So
$$\frac{\mathrm{d}}{\mathrm{d}t}[f \circ \boldsymbol{c}(t)] = \langle -2\sin t, 2\cos t, 1\rangle \cdot \langle -\sin t, \cos t, 1\rangle = 2\sin^2 t + 2\cos^2 t + 1 = 3$$

17. (§2.6) Given the real valued function $f(\boldsymbol{x})\colon \mathbb{R}^n \to \mathbb{R}$, and the vector $\boldsymbol{v}$, find the directional derivative of f along $\boldsymbol{v}$ at a given point, for

1. $f(x,y) = x^2 + y^2$, $\boldsymbol{v} = \langle 1, 3\rangle$, at the point $(3,4)$.
2. $f(x,y,z) = \sqrt{x^2+y^2+z^2}$, $\boldsymbol{v} = \langle -1, 0.5, 3\rangle$, at the point $(1,0,-1)$.
3. $f(x,y) = e^{x+y}$, $\boldsymbol{v} = \left\langle \sqrt{2}/2, \sqrt{2}/2\right\rangle$, at the point $(1,-1)$.
4. $f(x,y,z) = e^{x+y}\cos z + y^2$, $\boldsymbol{v} = \left\langle 1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}\right\rangle$, at the point $(2,1,\pi/2)$.
5. $f(x,y,z) = x + y\ln z$, $\boldsymbol{v} = \langle 1/3, 1, -1\rangle$ at the point $(1,2,0)$.

Answer: By directional derivative "along" $\boldsymbol{v}$, I mean $\nabla f \cdot \boldsymbol{v}$. When we talk about the directional derivative "in the direction of" $\boldsymbol{v}$, we mean $\nabla f \cdot (\boldsymbol{v}/\|\boldsymbol{v}\|)$. So these are easier to do these when you've already computed the gradient, as we have for many of these functions in problem 9.

1. $\nabla f(x,y) = \langle 2x, 2y\rangle$, and $\nabla f(3,4) = \langle 6, 8\rangle$, so the directional derivative is $\langle 6,8\rangle \cdot \langle 1,3\rangle = 6 + 24 = 30$. The meaning of this number is that a particle at $(3,4)$ moving with velocity $\boldsymbol{v}$ experiences an instantaneous change of f of 30 units/second.
2. $\nabla f(x,y,z) = \langle 2x, 2y, 2z\rangle$, and $\nabla f(1,0,-1) = \langle 2, 0, -2\rangle$, so the directional derivative is $\langle 2,0,-2\rangle \cdot \langle -1, 0.5, 3\rangle = -2 + 6 = 4$.
3. $\nabla f(x,y) = \langle e^{x+y}, e^{x+y}\rangle$, and $\nabla f(1,-1) = \langle 1,1\rangle$, so the directional derivative along $\boldsymbol{v}$ is $\langle 1,1\rangle \cdot \left\langle \sqrt{2}/2, \sqrt{2}/2\right\rangle = \sqrt{2}$.
4. $\nabla f(x,y,z) = \langle e^{x+y}\cos z, e^{x+y}\cos z + 2y, -e^{x+y}\sin z\rangle$, and $\nabla f(2,1,\pi/2) = \left\langle 0, 2, -e^3\right\rangle$, so the directional derivative along $\boldsymbol{v}$ is $\left\langle 0,2,-e^3\right\rangle \cdot \left\langle 1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}\right\rangle = \left(2 - e^3\right)/\sqrt{3}$

18. (§2.6) Given the real valued function $f(\boldsymbol{x})\colon \mathbb{R}^n \to \mathbb{R}$, and a point in $\mathbb{R}^n$, find the direction (*i.e.*, unit vector) of maximal increase of f at that point for

1. $f(x,y) = x^2 + y^2$, at the point $(3,4)$.
2. $f(x,y) = xy + 1/(xy)$, at the point $(1,2)$.
3. $f(x,y,z) = xyz$, at the point $(1,0,2)$.
4. $f(x,y,z) = 2x + 3y - 4z$, at the point $(1,-1,5)$.
5. $f(x,y,z) = x^2 + y^2 - z$, at the point $(0,1,1)$.

Answer: The "direction of maximal increase of f" is a vector in the same direction as ∇f. So this becomes a problem of computing the gradient and then making a unit length vector in the same direction.

1. $\nabla f(x,y) = \langle 2x, 2y\rangle$, and $\nabla f(3,4) = \langle 6,8\rangle$, so the direction of maximal increase is $\langle 6,8\rangle/\sqrt{36+64} = \langle 6/10, 8/10\rangle$.
2. $\nabla f(x,y) = \left\langle y - 1/(x^2y), x - 1/(xy^2)\right\rangle$, and $\nabla f(1,2) = \langle 3/2, 3/4\rangle$ The direction of maximal increase, then, is $\langle 3/2, 3/4\rangle/\sqrt{9/4+9/16} = (4/\sqrt{45})\langle 3/2, 3/4\rangle = \left(1/\sqrt{45}\right)\langle 6, 3\rangle$.

3. $\nabla f(x,y,z) = \langle yz, xz, xy\rangle$, and $\nabla f(1,0,2) = \langle 0,2,0\rangle$ so the direction of maximal increase is $\langle 0,1,0\rangle$.

19. (§3.2) Find the linear Taylor formula (*i.e.*, linearization) of $f(x,y) = x^2 + y^2$ at the point $(1,2)$.

Answer: We have

$$\begin{aligned} f\left((1,2)+(h_1,h_2)\right) &\approx f\left(1,2\right) + \nabla f\left(1,2\right)\cdot\langle h_1,h_2\rangle \\ &= \left(1^2+2^2\right) + \langle 2,4\rangle\cdot\langle h_1,h_2\rangle \\ &= 5 + 2h_1 + 4h_2 \end{aligned}$$

You can turn this into an equation involving (x,y) by letting $x = 1+h_1$, and $y = 2+h_2$. This gives

$$f\left(x,y\right) \approx 2x + 4y - 10,$$

the graph of which is a plane–the tangent plane of $f(x,y)$ at $(1,2)$.

20. (§3.2) Find the quadratic Taylor formula of $f(x,y) = x^2 + y^2$ at the point $(1,2)$.

Answer: We add a quadratic term on to the linear formula:

$$\begin{aligned} f\left((1,2)+(h_1,h_2)\right) &\approx f\left(1,2\right) + \nabla f\left(1,2\right)\cdot\langle h_1,h_2\rangle \\ &\quad + \frac{1}{2}\left[\frac{\partial^2 f}{\partial x^2}\left(1,2\right)h_1^2 + \frac{\partial^2 f}{\partial x\partial y}\left(1,2\right)h_1h_2 + \frac{\partial^2 f}{\partial y\partial x}\left(1,2\right)h_2h_1 + \frac{\partial^2 f}{\partial y^2}\left(1,2\right)h_2^2\right] \\ &= 5 + 2h_1 + 4h_2 + \frac{1}{2}\left[2h_1^2 + 0h_1h_2 + 0h_2h_1 + 2h_2^2\right] \\ &= 5 + 2h_1 + 4h_2 + h_1^2 + h_2^2. \end{aligned}$$

21. (§4.2) Given path $\boldsymbol{c}(t)$, find the arc length of the path for t between t_0 and t_1, for

1. $\boldsymbol{c}(t) = \langle 1+3t, 3, 1-4t\rangle$, $t_0 = 0$, $t_1 = 3$.
2. $\boldsymbol{c}(t) = \langle \cos t, \sin t, 3t\rangle$, $t_0 = 0$, $t_1 = 4\pi$.
3. $\boldsymbol{c}(t) = \left\langle t/\sqrt{2}, t/\sqrt{2}, \ln(\sec t)\right\rangle$, $t_0 = 0$, $t_1 = \pi/3$.
4. $\boldsymbol{c}(t) = \left\langle t, t^3/3 + 1/(4t)\right\rangle$, $t_0 = 1$, $t_1 = 2$.

Answer: Solveable problems of this type are few and far between. These examples pretty much exhaust my supply of good problems. We examined most of these paths in problem 14. Thus we don't have to recompute the speed. Recall that the arc length of $\boldsymbol{c}(t)$ for $t_0 \le t \le t_1$ is

$$\int_{t_0}^{t_1} \left\|\boldsymbol{c}'(t)\right\|\,\mathrm{d}t$$

1. $\int_0^3 5\,\mathrm{d}t = 15$
2. $\int_0^{4\pi} \sqrt{10}\,\mathrm{d}t = 4\pi\sqrt{10}$.
3. $\int_0^{\pi/3} \sec t\,\mathrm{d}t = \ln\left(\sec t + \cos t\right)\Big|_0^{\pi/3} = 2 + \sqrt{3} - 1 = 1 + \sqrt{3}$.

4. This one didn't appear correctly in problem 14, so we compute $\boldsymbol{c}'(t) = \langle 1, t^2 - 1/(4t^2) \rangle$. Now note that

$$\begin{aligned}\|\boldsymbol{c}'(t)\| &= \sqrt{1 + (t^2 - 1/(4t^2))^2} = \sqrt{1 + t^4 - 1/2 + 1/(16t^4)} \\ &= \sqrt{t^4 + 1/2 + 1/(16t^4)} = \sqrt{(t^2 + 1/(4t^2))^2} = t^2 + 1/(4t^2)\end{aligned}$$

Thus the answer is

$$\int_1^2 t^2 + 1/(4t^2)\,\mathrm{d}t = t^3/3 - 1/(4t)\Big|_1^2 = 8/3 - 1/8 - (1/3 - 1/4) = 7/3 + 1/8$$

22. (§4.4) Match four of the following vector fields to their graphical representation in Figure 1. (Two of the following fields are not plotted.)

1. $\boldsymbol{F}(x, y) = \langle y, x \rangle$.
2. $\boldsymbol{F}(x, y) = \langle x^2, y^2 \rangle$.
3. $\boldsymbol{F}(x, y) = \langle 2, 5 \rangle$.
4. $\boldsymbol{F}(x, y) = \langle y, -x \rangle$.
5. $\boldsymbol{F}(x, y) = \langle 2x, 2y \rangle$.
6. $\boldsymbol{F}(x, y) = \langle -y, x \rangle$.

Answer: One way to do this is to sketch each of these vector fields. By "sketch" I mean pick a few points and compute the vector field at those points and see what you get. You will find that (a) is $\langle 2x, 2y \rangle$, (b) is $\langle 2, 5 \rangle$, (c) is $\langle y, x \rangle$, and (d) is $\langle -y, x \rangle$.

23. (§4.4) Draw some flow lines for the vector fields in Figure 1.

Answer: You do that.

24. (§4.4) Which of the vector fields plotted in Figure 1 might be gradient fields, that is of the form $\nabla\phi$ for some scalar field ϕ? For those that are, try to draw some level sets for the function ϕ over the vector field.

Answer: In (a) the field is the gradient of $\phi(x, y) = x^2 + y^2$. The level sets are concentric circles centered at the origin. In (b) the field is the gradient of $\phi(x, y) = 2x + 5y$. The level sets are straight lines with slope $-2/5$. In (c) the field is the gradient of $\phi(x, y) = xy$. The level sets are those hyperbola like things (see my lame description of these in problem 7). The field in (d) is not a gradient. If you try to draw a level set you will find they are lines going through the origin, and thus they all have the same value and thus ϕ has to be a constant and its gradient would have to be the field of all zeros, a contradiction.

25. (§4.4) Given the vector field $\boldsymbol{F}(x, y, z)$, find div $\boldsymbol{F} = \nabla \cdot \boldsymbol{F}$, and $\mathbf{curl}\,\boldsymbol{F} = \nabla \times \boldsymbol{F}$ for

1. $\boldsymbol{F}(x, y, z) = \langle xy + z, x^2 - y, z^2 - 3 \rangle$.
2. $\boldsymbol{F}(x, y, z) = \langle yz, xz, xy \rangle$.
3. $\boldsymbol{F}(x, y, z) = \langle 2x, 2y, -1 \rangle$.
4. $\boldsymbol{F}(x, y, z) = \langle y, -x, 0 \rangle$.
5. $\boldsymbol{F} = \nabla\phi$, where $\phi(x, y, z) = 2xy^2 - ze^y$.
6. $\boldsymbol{F} = \nabla \times \boldsymbol{G}$, where $\boldsymbol{G}(x, y, z) = \langle xy + z, x^2 - y, z^2 - 3 \rangle$.

Answer: 1. $\boldsymbol{F} = \left\langle xy + z, x^2 - y, z^2 - 3 \right\rangle$: $\nabla \cdot \boldsymbol{F}(x, y, z) = y + -1 + 2z$. $\nabla \times \boldsymbol{F}(x, y, z) = \langle 0, 1, x \rangle$

2. $\boldsymbol{F} = \langle yz, xz, xy \rangle$: $\nabla \cdot \boldsymbol{F} = 0$. $\nabla \times \boldsymbol{F} = \mathbf{0}$.
3. $\boldsymbol{F} = \langle 2x, 2y, -1 \rangle$: $\nabla \cdot \boldsymbol{F} = 4$. $\nabla \times \boldsymbol{F} = \mathbf{0}$.
4. $\boldsymbol{F} = \langle y, -x, 0 \rangle$: $\nabla \cdot \boldsymbol{F} = 0$. $\nabla \times \boldsymbol{F} = \langle 0, 0, -2 \rangle$.
5. $\boldsymbol{F} = \nabla \phi$, where $\phi(x, y, z) = 2xy^2 - ze^y$: in this case $\nabla \cdot \boldsymbol{F} = \nabla \cdot \nabla \phi$. This is sometimes written as $\nabla^2 \phi$, or, more confusingly, as $\Delta \phi$. This is known as the *Laplacian* of ϕ. It turns out to be the sum of the second partials. That is $\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$. In this case we get $\nabla^2 \phi = 0 + (4x - ze^y) + 0$. There was nothing stopping you, however, from first evaluating $\boldsymbol{F}$, then taking its divergence–you should get the same answer. The curl of this $\boldsymbol{F}$ is $\mathbf{0}$ because we have the rule that $\nabla \times \nabla \phi = \mathbf{0}$ for ϕ a function with continuous second derivatives.
6. $\boldsymbol{F} = \nabla \times \boldsymbol{G}$, where $\boldsymbol{G}(x, y, z) = \left\langle xy + z, x^2 - y, z^2 - 3 \right\rangle$.
   We have a rule that $\nabla \cdot (\nabla \times \boldsymbol{G}) = 0$ for a field whose components have continuous second partial derivatives, like this one. So the divergence is zero. Alternatively we could evaluate $\boldsymbol{F}$ and then take its divergence. We did this above in an earlier part of this problem, and have $\boldsymbol{F} = \langle 0, 1, x \rangle$. Taking the divergence indeed gives 0. Now to the curl we evaluate $\nabla \times \langle 0, 1, x \rangle = \langle 0, 1, 0 \rangle$.

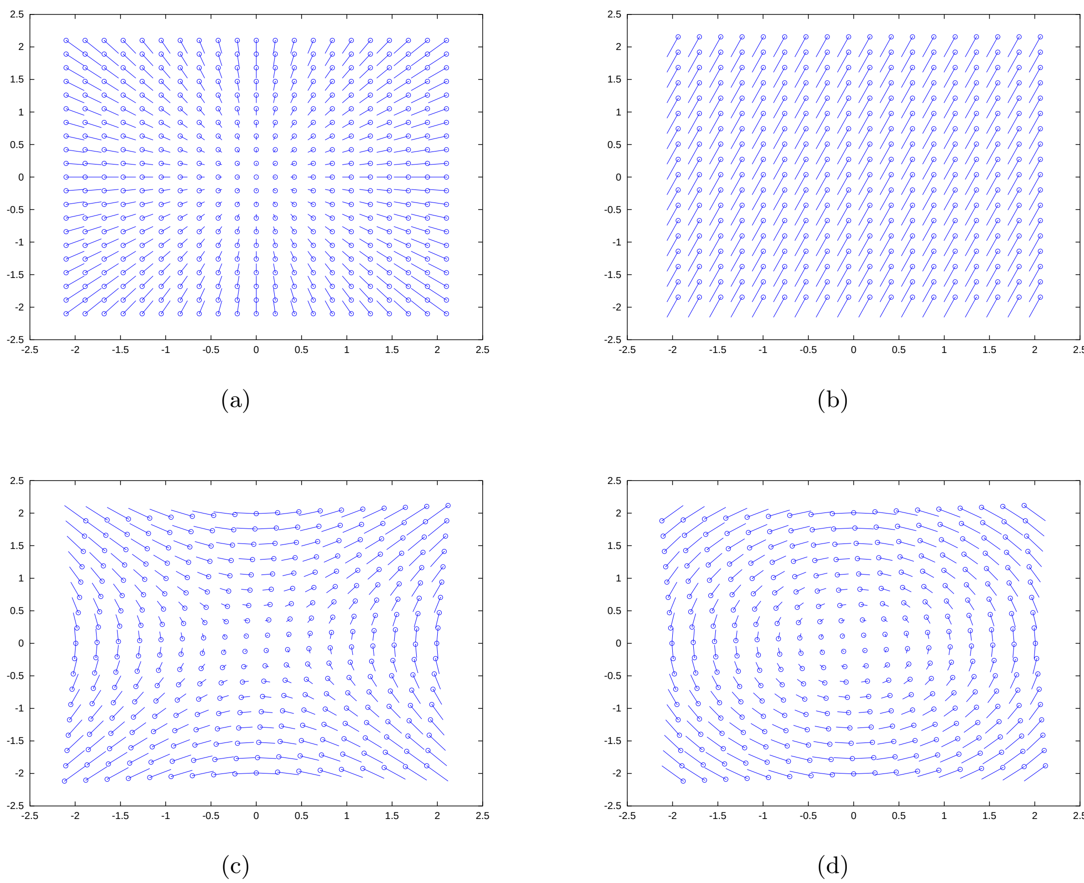

Figure 1: Four vector fields. Circles represent the tips of the vectors (proper arrowheads were not available).