Exam 2 Preparation

Exam 2 is Monday, February 28$^{\text{th}}$, and covers §5.1–5.5, 6.1, 6.2, 7.1–7.6. *Please bring a blue book for the exam.* You may not use a calculator or notes. The following formulæ will be provided on your exam:

$$\mathrm{d}\boldsymbol{s} = \boldsymbol{c}'(t)\,\mathrm{d}t \qquad \mathrm{d}\boldsymbol{S} = (\boldsymbol{T}_u \times \boldsymbol{T}_v)\,\mathrm{d}u\,\mathrm{d}v = \left(\frac{\partial \boldsymbol{\Phi}}{\partial u} \times \frac{\partial \boldsymbol{\Phi}}{\partial v}\right)\mathrm{d}u\,\mathrm{d}v \qquad \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

Everything else must be committed to memory.

What follows are questions similar to your homework assignments. You should be prepared to answer at least questions like these. This list may not be exhaustive.

1. (§5.1–5.4) Given function $f(x,y)$ and region $D \subseteq \mathbb{R}^2$, evaluate

$$\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

1. $f(x,y) = 2xy$, D is the region bounded by $y = 0, x = 2, y = x^2$.
2. $f(x,y) = y^2 \cos x$, D is region bounded by $x = y^3, y = -1, y = 1, x = 3$.

2. (§5.1–5.4) Given function $f(x,y,z)$ and region $D \subseteq \mathbb{R}^3$, evaluate

$$\iiint_D f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

1. $f(x,y,z) = \sqrt{x^2+z^2}$, D is bounded by $y = x^2 + z^2, y = 4$.
2. $f(x,y,z) = x + y + z$, D is bounded by $x = y^2, x = z, z = 0, x = 1$.

3. (§6.1) Given a change of variables transformation $T\colon D^* \subseteq \mathbb{R}^n \to \mathbb{R}^n$, and the domain, D^*, find the image of the transformation, $D = T(D^*)$. Is the transformation one-to-one?

1. $(x,y) = T(r,\theta) = (r\cos\theta, r\sin\theta)$, and $D^* = [0,4] \times [0,\pi/2]$.
2. $(x,y,z) = T(r,\theta,z) = (r\cos\theta, r\sin\theta, z)$, and $D^* = [0,1] \times [0,\pi] \times [0,1]$.

4. (§6.1) Given a change of variables transformation $T\colon D^* \subseteq \mathbb{R}^n \to \mathbb{R}^n$, and a set D, find a domain, D^*, such that $D = T(D^*)$. Is the transformation one-to-one?

1. $(x,y) = T(u,v) = (u/v, v)$, and D is the region bounded by the curves $xy = 1, xy = 4, y = 1, y = 3$.

5. (§6.2) Given a change of variables transformation $T\colon D^* \subseteq \mathbb{R}^n \to \mathbb{R}^n$, find the Jacobian Determinant of the transformation.

1. With $(x,y) = T(r,\theta) = (r\cos\theta, r\sin\theta)$, find $\frac{\partial(x,y)}{\partial(r,\theta)}$.
2. With $(x,y) = T(u,v) = (u/v, v)$, find $\frac{\partial(x,y)}{\partial(u,v)}$.
3. With $(x,y) = T(u,v) = ((u+v)/2, (u-v)/2)$, find $\frac{\partial(x,y)}{\partial(u,v)}$.
4. With $(x,y) = T(u,v) = (au + bv + c, du + fv + g)$, find $\frac{\partial(x,y)}{\partial(u,v)}$.
5. With $(x,y,z) = T(u,v,w) = (au, bv, cw)$, find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$.
6. With $(x,y,z) = T(r,\theta,z) = (r\cos\theta, r\sin\theta, z)$, find $\frac{\partial(x,y,z)}{\partial(r,\theta,z)}$.

7. With $(x,y,z) = T(\rho,\theta,\phi) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$, find $\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}$

6. (§6.2) Given a change of variables transformation $T\colon D^* \subseteq \mathbb{R}^n \to \mathbb{R}^n$, some set D, and a function $f(x,y)$, evaluate

$$\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y \quad \text{as an integral of the form} \quad \iint_{D^*} f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\,\mathrm{d}v$$

You may be forced to find D^* yourself.

1. $f(x,y) = \sin\left(x^2 + 2xy + y^2\right)$, D is the triangle with corners $(0,0)$,$(1,0)$,$(0,1)$ using the change of variables $x = (u+v)/2, y = (u-v)/2$.
2. $f(x,y) = e^{xy}$, D is bounded by $xy = 1, xy = 4, y = 1, y = 3$, and using the change of variables $x = u/v, y = v$.

7. (§7.1) Given the scalar field $f(x,y,z)$ and path $\boldsymbol{c}(t)\colon [a,b] \to \mathbb{R}^3$ evaluate the path integral

$$\int_{\boldsymbol{c}} f\,\mathrm{d}s$$

1. $f(x,y,z) = x^2 + y^2$, along $\boldsymbol{c}(t) = \langle r\cos t, r\sin t, t\rangle$ for $a \le t \le b$.
2. $f(x,y,z) = z$, along $\boldsymbol{c}(t) = \langle \cos t, \sin t, t\rangle$ for $0 \le t \le 2\pi$.
3. $f(x,y,z) = \cos\left(\sqrt{z} + \sqrt{y}\right)$, along $\boldsymbol{c}(t) = \left\langle 1, t^2/4, t^2/4\right\rangle$ for $0 \le t \le 1$.

8. (§7.2) Given the vector field $\boldsymbol{F}(x,y,z)$ and path $\boldsymbol{c}(t)\colon [a,b] \to \mathbb{R}^3$ evaluate the line integral

$$\int_{\boldsymbol{c}} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s}$$

1. $\boldsymbol{F} = \left\langle 3x + 4y, 2x + 3y^2\right\rangle$, along $\boldsymbol{c}(t) = \langle 2\cos t, 2\sin t\rangle$, $0 \le t \le 2\pi$.
2. $\boldsymbol{F} = \left\langle 2xy, x^2 + z, y\right\rangle$, along the path which traces a straight line from $(1,0,2)$ to $(3,4,1)$.
3. $\boldsymbol{F} = \langle yz, xz, xy\rangle$ along some path $\boldsymbol{c}$ with $\boldsymbol{c}(a) = (-1,-1,2)$, and $\boldsymbol{c}(b) = (-2,1,-1)$.
4. $\boldsymbol{F} = \left\langle 2xz + y^2z^2, 2xyz^2, x^2 + 2xy^2z\right\rangle$ along the path $\boldsymbol{c}(t) = \left\langle \cos(\pi t) + e^t, -\sin(\pi t) + \log(1+t), t^{12}\right\rangle$ for $0 \le t \le 1$.

9. (§7.2) Given a vector field $\boldsymbol{F}$, determine if it is conservative. If it is conservative, find ϕ such that $\boldsymbol{F} = \nabla\phi$.

1. $\boldsymbol{F} = \left\langle 2xy, x^2 + z, y\right\rangle$.
2. $\boldsymbol{F} = \langle xy, z, x\rangle$.
3. $\boldsymbol{F} = \left\langle x^2y + 1, \frac{1}{3}x^3 + 1, y\right\rangle$.
4. $\boldsymbol{F} = \frac{1}{x^2+y^2+z^2}\langle x, y, z\rangle$.
5. $\boldsymbol{F} = \langle -\sin x\cos y, \cos x\cos y, 1\rangle$.
6. $\boldsymbol{F} = \frac{1}{(x^2+y^2+z^2)^{3/2}}\langle x, y, z\rangle$.

10. (§7.3) Given a parametrization of a surface, $\mathbf{\Phi}(u, v)$, find $\boldsymbol{T}_u \times \boldsymbol{T}_v$. Is the parametrization regular?

1. $\mathbf{\Phi}(u, v) = \left\langle v \cos u, v \sin u, v^2 \right\rangle$.
2. $\mathbf{\Phi}(u, v) = \langle (1 + \cos u) \cos v, (1 + \cos u) \sin v, \sin v \rangle$.
3. $\mathbf{\Phi}(u, v) = \langle u, v, g(u, v) \rangle$.
4. $\mathbf{\Phi}(u, v) = \left\langle u, v, u^2 + v^{3/2} \right\rangle$.
5. $\mathbf{\Phi}(u, v) = \left\langle u^2, v - u, vu \right\rangle$.

11. (§7.4) Given some surface, S, find a regular parametrization $\mathbf{\Phi}(u, v)$, and domain $D \subseteq \mathbb{R}^2$ such that $S = \mathbf{\Phi}(D)$. Find $\mathrm{d}\boldsymbol{S}$ and $\mathrm{d}S = \| \mathrm{d}\boldsymbol{S} \|$. Set up an integral for the surface area of S.

1. S is the part of the cone $x^2 = y^2 + z^2$ inside the sphere $(x - 8)^2 + y^2 + z^2 = 49$.
2. S is the part of the paraboloid $z = x^2 + y^2$ above the xy plane and below the plane $x + y + z = 9$.

12. (§7.5) Given some oriented surface, S, parametrized by $\mathbf{\Phi}$, and some scalar field f, find the integral of f over S:

$$\iint_S f \, \mathrm{d}S$$

1. $f(x, y, z) = 3x^2$ over the sphere of radius r.
2. $f(x, y, z) = 240xy$ over S, which is the paraboloid $z = x^2 + y^2$ for $x \in [0, 1]$, $y \in [0, 1]$.
3. $f(x, y, z) = yz$ over S, which is the boundary of the cube $[0, 1] \times [0, 1] \times [0, 1]$. That is, the cube with corners $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 1, 0)$, $(0, 1, 1)$, $(1, 0, 1)$, $(1, 1, 1)$.
4. $f(x, y, z) = 5$ over S, which is the plane $2x + 3y + 6z - 2 = 0$, with $x \geq 0$, $y \geq 0$, $z \geq 0$.

13. (§7.6) Given some oriented surface, S, parametrized by $\mathbf{\Phi}$, and some vector field $\boldsymbol{F}$, find the surface integral of $\boldsymbol{F}$,

$$\iint_S \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S}$$

1. $\boldsymbol{F} = \left\langle x^2, y, z^3 \right\rangle$, S is the surface of the cube bounded by $x = 0, x = 2, y = \pm 1, z = \pm 1$. Use outward normals.
2. $\boldsymbol{F} = \langle 0, 0, \cos(xy + 2^z) \rangle$, S is the part of the cylinder $x^2 + y^2 = 1$ with $0 \leq z \leq 2$, using the usual parametrization and an outward normal.
3. $\boldsymbol{F} = \langle y + z, 2x + y, y \rangle$, S is the surface of the triangle with corners $(1, 0, 0)$, $(0, 2, 0)$, $(0, 0, 3)$ with normal pointing away from the origin.
4. $\boldsymbol{F} = \langle x, y, -z \rangle$, S is the surface of the ellipsoid $x^2 + \frac{1}{4}y^2 + \frac{1}{9}z^2 = 9$. Use the parametrization $\mathbf{\Phi}(\phi, \theta) = (3 \sin\phi \cos\theta, 6 \sin\phi \sin\theta, 9 \cos\phi)$.
5. $\boldsymbol{F} = \langle 2y, 2x, z \rangle$, S is the part of the cone $x^2 = y^2 + z^2$ inside the sphere $(x - 8)^2 + y^2 + z^2 = 49$. Assume $\boldsymbol{n} \cdot \boldsymbol{i} < 0$.
6. $\boldsymbol{F} = \left\langle 3, x^2, y \right\rangle$, S is the part of the paraboloid $z = x^2 + y^2$ above the xy plane and below the plane $x + y + z = 9$. Assume $\boldsymbol{n} \cdot \boldsymbol{k} > 0$.

14. (misc.) Let C be a curve representing the intersection of the sphere $x^2 + y^2 + z^2 = r^2$ and the plane $x + y + z = 0$. Let $f(x, y, z) = x^2$ be the mass density of a wire running along C. What is the total mass of the wire.

15. (misc.) Let $\phi(x, y, z) = e^x + yz$ represent the electric potential in space. What is the work done by the field $-\nabla\phi$ on a particle which moves from $(0, 2, 3)$ to $(3, -2, 1)$.