Exam 2 Preparation

Exam 2 is Monday, February 28$^{\text{th}}$, and covers §5.1–5.5, 6.1, 6.2, 7.1–7.6. *Please bring a blue book for the exam.* You may not use a calculator or notes. The following formulæ will be provided on your exam:

$$\mathrm{d}\boldsymbol{s} = \boldsymbol{c}'(t)\,\mathrm{d}t \qquad \mathrm{d}\boldsymbol{S} = \boldsymbol{T}_u \times \boldsymbol{T}_v\,\mathrm{d}u\,\mathrm{d}v = \left(\frac{\partial \boldsymbol{\Phi}}{\partial u} \times \frac{\partial \boldsymbol{\Phi}}{\partial v}\right)\mathrm{d}u\,\mathrm{d}v \qquad \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

Everything else must be committed to memory.

What follows are questions similar to your homework assignments. You should be prepared to answer at least questions like these. This list may not be exhaustive.

1. (§5.1–5.4) Given function $f(x,y)$ and region $D \subseteq \mathbb{R}^2$, evaluate

$$\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

1. $f(x,y) = 2xy$, D is the region bounded by $y = 0, x = 2, y = x^2$.
2. $f(x,y) = y^2 \cos x$, D is region bounded by $x = y^3, y = -1, y = 1, x = 3$.

Answer: These problems are really review from 20C.

1. I get

$$\int_0^2 \int_0^{x^2} 2xy\,\mathrm{d}y\,\mathrm{d}x = 32/3$$

2. This is

$$\int_{-1}^{1} \int_{y^3}^{3} y^2 \cos x\,\mathrm{d}x\,\mathrm{d}y = \cos(1)$$

2. (§5.1–5.4) Given function $f(x,y,z)$ and region $D \subseteq \mathbb{R}^3$, evaluate

$$\iiint_D f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

1. $f(x,y,z) = \sqrt{x^2+z^2}$, D is bounded by $y = x^2 + z^2, y = 4$.
2. $f(x,y,z) = x + y + z$, D is bounded by $x = y^2, x = z, z = 0, x = 1$.

Answer:
1. Here the region D is a paraboloid, but because the coordinates have been switched, it seems more dificult. I think the answer is $\frac{256\pi}{45}\sqrt{2}$.
2. It is a bit tricky to visualize D. My answer was 6/7.

3. (§6.1) Given a change of variables transformation $T\colon D^* \subseteq \mathbb{R}^n \to \mathbb{R}^n$, and the domain, D^*, find the image of the transformation, $D = T(D^*)$. Is the transformation one-to-one?

1. $(x,y) = T(r,\theta) = (r\cos\theta, r\sin\theta)$, and $D^* = [0,4] \times [0,\pi/2]$.

2. $(x,y,z) = T(r,\theta,z) = (r\cos\theta, r\sin\theta, z)$, and $D^* = [0,1] \times [0,\pi] \times [0,1]$.

Answer: 1. D is the part of a disc of radius 4 in the first quadrant. The mapping is not one-to-one on D^* because $T(0,\theta) = (0,0)$ for any θ. However, the mapping *is* one-to-one on the interior of D^*. By "interior," I mean $(0,4) \times (0,\pi/2)$.

2. D is half of a solid cylinder of radius 1 and height 1. It is the part of this cylinder in the halfspace $y \geq 0$. Again, the mapping is not one-to-one on D^* because it takes many points to the z-axis.

4. (§6.1) Given a change of variables transformation $T\colon D^* \subseteq \mathbb{R}^n \to \mathbb{R}^n$, and a set D, find a domain, D^*, such that $D = T(D^*)$. Is the transformation one-to-one?

1. $(x,y) = T(u,v) = (u/v, v)$, and D is the region bounded by the curves $xy = 1, xy = 4, y = 1, y = 3$.

Answer: 1. A good way to find the "preimage" of a set under a transformation is to "pull-back" the equation of its boundaries. In this case, the region is bounded by the curves $xy = 1, xy = 4, y = 1, y = 3$. If we write these in terms of u and v, we will find curves which bound D^*. Thus under the transfrom $(x,y) = (u/v, v)$, the equation $xy = 1$ becomes $(u/v)v = 1$, or $u = 1$. The three other curves are handled similarly to give $D^* = [1,4] \times [1,3]$. This transformation is one-to-one.

5. (§6.2) Given a change of variables transformation $T\colon D^* \subseteq \mathbb{R}^n \to \mathbb{R}^n$, find the Jacobian Determinant of the transformation.

1. With $(x,y) = T(r,\theta) = (r\cos\theta, r\sin\theta)$, find $\frac{\partial(x,y)}{\partial(r,\theta)}$.
2. With $(x,y) = T(u,v) = (u/v, v)$, find $\frac{\partial(x,y)}{\partial(u,v)}$.
3. With $(x,y) = T(u,v) = ((u+v)/2, (u-v)/2)$, find $\frac{\partial(x,y)}{\partial(u,v)}$.
4. With $(x,y) = T(u,v) = (au + bv + c, du + fv + g)$, find $\frac{\partial(x,y)}{\partial(u,v)}$.
5. With $(x,y,z) = T(u,v,w) = (au, bv, cw)$, find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$.
6. With $(x,y,z) = T(r,\theta,z) = (r\cos\theta, r\sin\theta, z)$, find $\frac{\partial(x,y,z)}{\partial(r,\theta,z)}$.
7. With $(x,y,z) = T(\rho,\theta,\phi) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$, find $\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}$

Answer: Remember that

$$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

1. $\frac{\partial(x,y)}{\partial(r,\theta)} = r.$
2. $\frac{\partial(x,y)}{\partial(u,v)} = 1/v.$
3. $\frac{\partial(x,y)}{\partial(u,v)} = -1.$
4. $\frac{\partial(x,y)}{\partial(u,v)} = af - bd.$
5. $\frac{\partial(x,y,z)}{\partial(u,v,w)} = abc.$
6. $\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = r.$
7. $\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \rho^2 \sin\phi$

6. (§6.2) Given a change of variables transformation $T\colon D^* \subseteq \mathbb{R}^n \to \mathbb{R}^n$, some set D, and a function $f(x,y)$, evaluate

$$\iint\limits_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y \quad \text{as an integral of the form} \quad \iint\limits_{D^*} f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\,\mathrm{d}v$$

You may be forced to find D^* yourself.

1. $f(x,y) = \sin\left(x^2+2xy+y^2\right)$, D is the triangle with corners $(0,0)$,$(1,0)$,$(0,1)$ using the change of variables $x = (u+v)/2, y = (u-v)/2$.
2. $f(x,y) = e^{xy}$, D is bounded by $xy = 1, xy = 4, y = 1, y = 3$, and using the change of variables $x = u/v, y = v$.

Answer: 1. The set D is bounded by $x = 0$, $y = 0$, and $y = 1-x$. Using the given transformation, these become $u+v = 0$, $u-v = 0$, and $u = 1$. The integral becomes

$$\iint\limits_{D^*} \sin\left(u^2\right)\left|-\frac{1}{2}\right| \mathrm{d}A = \frac{1}{2}\int_0^1\int_{-u}^{u} \sin\left(u^2\right)\,\mathrm{d}v\,\mathrm{d}u = -\frac{1}{2}\cos(1)+\frac{1}{2}$$

2. See Problem 4 to find D^*.
$\left(e^4-e\right)\ln 3$.

7. (§7.1) Given the scalar field $f(x,y,z)$ and path $\boldsymbol{c}(t)\colon [a,b] \to \mathbb{R}^3$ evaluate the path integral

$$\int\limits_{\boldsymbol{c}} f\,\mathrm{d}s$$

1. $f(x,y,z) = x^2+y^2$, along $\boldsymbol{c}(t) = \langle r\cos t, r\sin t, t\rangle$ for $a \le t \le b$.
2. $f(x,y,z) = z$, along $\boldsymbol{c}(t) = \langle \cos t, \sin t, t\rangle$ for $0 \le t \le 2\pi$.
3. $f(x,y,z) = \cos\left(\sqrt{z}+\sqrt{y}\right)$, along $\boldsymbol{c}(t) = \left\langle 1, t^2/4, t^2/4\right\rangle$ for $0 \le t \le 1$.

Answer: There is no shortcut for path integrals.

1. First, $\|\boldsymbol{c}'(t)\| = \sqrt{r^2+1}$. So you get

$$\int_a^b r^2\sqrt{r^2+1}\,\mathrm{d}t = (b-a)\,r^2\sqrt{1+r^2}$$

2. I get $\int_0^{2\pi} t\sqrt{2}\,\mathrm{d}t = 2\sqrt{2}\pi^2$.
3. $\int_0^1 \cos t\left(t/\sqrt{2}\right)\,\mathrm{d}t$, which you solve by parts to get $(\sin 1+\cos 1-1)/\sqrt{2}$.

8. (§7.2) Given the vector field $\boldsymbol{F}(x,y,z)$ and path $\boldsymbol{c}(t)\colon [a,b] \to \mathbb{R}^3$ evaluate the line integral

$$\int\limits_{\boldsymbol{c}} \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{s}$$

1. $\boldsymbol{F} = \left\langle 3x+4y, 2x+3y^2\right\rangle$, along $\boldsymbol{c}(t) = \langle 2\cos t, 2\sin t\rangle$, $0 \le t \le 2\pi$.

2. $\boldsymbol{F} = \left\langle 2xy, x^2+z, y \right\rangle$, along the path which traces a straight line from $(1,0,2)$ to $(3,4,1)$.
3. $\boldsymbol{F} = \langle yz, xz, xy \rangle$ along some path $\boldsymbol{c}$ with $\boldsymbol{c}(a) = (-1,-1,2)$, and $\boldsymbol{c}(b) = (-2,1,-1)$.
4. $\boldsymbol{F} = \left\langle 2xz + y^2z^2, 2xyz^2, x^2 + 2xy^2z \right\rangle$ along the path $\boldsymbol{c}(t) = \left\langle \cos(\pi t) + e^t, -\sin(\pi t) + \log(1+t), t^{12} \right\rangle$ for $0 \le t \le 1$.

Answer: Some of these use the following fact (which is one of the reasons we care about conservative vector fields): if $\boldsymbol{F}$ is conservative, *i.e.*, $\boldsymbol{F} = \nabla\phi$ for a scalar field ϕ, then

$$\int_{\boldsymbol{c}} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = \int_a^b \nabla\phi(\boldsymbol{c}(t)) \cdot \boldsymbol{c}'(t)\,\mathrm{d}t = \phi(\boldsymbol{c}(t))\Big|_a^b = \phi(\boldsymbol{c}(b)) - \phi(\boldsymbol{c}(a))$$

We should use this shortcut when $\boldsymbol{c}$ is given ambiguously, or when the problem would be too horendous to do the long way.

1. We must do this directly, since this field is not conservative. The answer requires tedious calculations involving trigonometry:

$$\int_0^{2\pi} \left\langle 6\cos t + 8\sin t, 4\cos t + 12\sin^2 t \right\rangle \cdot \langle -2\sin t, 2\cos t \rangle \,\mathrm{d}t$$
$$= \int_0^{2\pi} -4\sin t\cos t - 16\sin^2 t + 24\sin^2 t\cos t\,\mathrm{d}t \ldots$$

2. 40. Can be done directly or by using a potential.
3. 0. The path is given only vaguely, so we must use the fact that the field is conservative: $\boldsymbol{F} = \nabla\phi$ with $\phi = xyz$.
4. This problem is so horrendous, you probably cannot do it except by using the potential of the field, $\phi = x^2z + xy^2z^2$. The answer is $(e-1)\left[e - 1 + (\log 2)^2\right]$.

9. (§7.2) Given a vector field $\boldsymbol{F}$, determine if it is conservative. If it is conservative, find ϕ such that $\boldsymbol{F} = \nabla\phi$.

1. $\boldsymbol{F} = \left\langle 2xy, x^2 + z, y \right\rangle$.
2. $\boldsymbol{F} = \langle xy, z, x \rangle$.
3. $\boldsymbol{F} = \left\langle x^2y + 1, \frac{1}{3}x^3 + 1, y \right\rangle$.
4. $\boldsymbol{F} = \frac{1}{x^2+y^2+z^2} \langle x, y, z \rangle$.
5. $\boldsymbol{F} = \langle -\sin x \cos y, \cos x \cos y, 1 \rangle$.
6. $\boldsymbol{F} = \frac{1}{(x^2+y^2+z^2)^{3/2}} \langle x, y, z \rangle$.

Answer:

1. $\phi = x^2y + zy$.
2. not conservative.
3. not conservative.
4. $\phi = \frac{1}{2}\ln\left(x^2 + y^2 + z^2\right)$.
5. not conservative.

6. $\phi = -\frac{1}{\sqrt{x^2+y^2+z^2}}$.

10. (§7.3) Given a parametrization of a surface, $\mathbf{\Phi}(u, v)$, find $\boldsymbol{T}_u \times \boldsymbol{T}_v$. Is the parametrization regular?

1. $\mathbf{\Phi}(u, v) = \left\langle v\cos u, v\sin u, v^2 \right\rangle$.
2. $\mathbf{\Phi}(u, v) = \langle (1+\cos u)\cos v, (1+\cos u)\sin v, \sin v\rangle$.
3. $\mathbf{\Phi}(u, v) = \langle u, v, g(u, v)\rangle$.
4. $\mathbf{\Phi}(u, v) = \left\langle u, v, u^2 + v^{3/2} \right\rangle$.
5. $\mathbf{\Phi}(u, v) = \left\langle u^2, v-u, vu \right\rangle$.

Answer:

1. $\boldsymbol{T}_u \times \boldsymbol{T}_v = \left\langle -2v^2\cos u, -2v^2\sin u, v \right\rangle$. This surface is regular where $v \neq 0$.
2. $\boldsymbol{T}_u \times \boldsymbol{T}_v = \left\langle -\sin u \sin v \cos v, \sin u \cos^2 v, (1+\cos u)\sin u \right\rangle$. This surface is regular where $\sin u \neq 0$.
3. $\boldsymbol{T}_u \times \boldsymbol{T}_v = \left\langle -\frac{\partial g}{\partial u}, -\frac{\partial g}{\partial v}, 1 \right\rangle$. This is regular where ∇g is defined.
4. $\boldsymbol{T}_u \times \boldsymbol{T}_v = \langle -2u, -3/2\sqrt{v}, 1\rangle$. This is regular for $v \geq 0$. If $v < 0$, then $\mathbf{\Phi}(u, v)$ is not defined, and neither is $\boldsymbol{T}_u \times \boldsymbol{T}_v$.
5. $\boldsymbol{T}_u \times \boldsymbol{T}_v = \left\langle -u-v, -2u^2, 2u \right\rangle$. This is regular except for $(u, v) = (0, 0)$.

11. (§7.4) Given some surface, S, find a regular parametrization $\mathbf{\Phi}(u, v)$, and domain $D \subseteq \mathbb{R}^2$ such that $S = \mathbf{\Phi}(D)$. Find $\mathrm{d}\boldsymbol{S}$ and $\mathrm{d}S = \|\mathrm{d}\boldsymbol{S}\|$. Set up an integral for the surface area of S.

1. S is the part of the cone $x^2 = y^2 + z^2$ inside the sphere $(x-8)^2 + y^2 + z^2 = 49$.
2. S is the part of the paraboloid $z = x^2 + y^2$ above the xy plane and below the plane $x + y + z = 9$.

Answer:

1. First look at the geometry of the situation: if (x, y, z) is "inside" the sphere $(x-8)^2 + y^2 + z^2 = 49$, then we should have $(x-8)^2 + y^2 + z^2 \leq 49$. Moreover, if (x, y, z) is on the surface of the cone, then $x^2 = y^2 + z^2$. Merging this into the inequality gives $(x-8)^2 + x^2 \leq 49$, *i.e.*, $2x^2 - 16x + 15 \leq 0$. This is a classic precalculus problem, with solution

$$4 - \sqrt{17/2} \leq x \leq 4 + \sqrt{17/2}$$

Now parametrize in terms of r and θ (we could call them u and v, but r and θ should be more familiar). This gives $\mathbf{\Phi}(r, \theta) = \langle r, r\cos\theta, r\sin\theta\rangle$, for the region

$$D = \left\{ (r, \theta) \,\middle|\, 4 - \sqrt{17/2} \leq r \leq 4 + \sqrt{17/2}, 0 \leq \theta \leq 2\pi \right\}.$$

We have $\mathrm{d}\boldsymbol{S} = \langle r, -r\cos\theta, -r\sin\theta\rangle \,\mathrm{d}r\,\mathrm{d}\theta$, and $\mathrm{d}S = \sqrt{2}r\,\mathrm{d}r\,\mathrm{d}\theta$. The surface area is

$$A(S) = \iint_D \mathrm{d}S = \int_0^{2\pi} \int_{4-\sqrt{17/2}}^{4+\sqrt{17/2}} \sqrt{2}r\,\mathrm{d}r\,\mathrm{d}\theta = 16\pi\sqrt{17}$$

2. We did something like this in class. We have $\mathbf{\Phi}(u, v) = \left\langle u, v, u^2 + v^2 \right\rangle$, with

$$D = \left\{ (u, v) \,\middle|\, (u + 1/2)^2 + (v + 1/2)^2 = 19/2 \right\}$$

This is the graph of a function so we have $\mathrm{d}\boldsymbol{S} = \langle -2u, -2v, 1\rangle \,\mathrm{d}u\,\mathrm{d}v$, and $\mathrm{d}S = \sqrt{1+4u^2+4v^2}\,\mathrm{d}u\,\mathrm{d}v$. We calculate the surface area as

$$A(S) = \iint_D \mathrm{d}S = \iint_D \sqrt{1+4u^2+4v^2}\,\mathrm{d}u\,\mathrm{d}v.$$

This is a hard integral.

12. (§7.5) Given some oriented surface, S, parametrized by $\boldsymbol{\Phi}$, and some scalar field f, find the integral of f over S:

$$\iint_S f\,\mathrm{d}S$$

1. $f(x,y,z) = 3x^2$ over the sphere of radius r.
2. $f(x,y,z) = 240xy$ over S, which is the paraboloid $z = x^2+y^2$ for $x \in [0,1]$, $y \in [0,1]$.
3. $f(x,y,z) = yz$ over S, which is the boundary of the cube $[0,1]\times[0,1]\times[0,1]$. That is, the cube with corners $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,1,0)$, $(0,1,1)$, $(1,0,1)$, $(1,1,1)$.
4. $f(x,y,z) = 5$ over S, which is the plane $2x+3y+6z-2=0$, with $x \geq 0$, $y \geq 0$, $z \geq 0$.

Answer:

1. Using the parametrization $\boldsymbol{\Phi}(\phi,\theta) = \langle r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi\rangle$, we get

$$\boldsymbol{T}_\phi \times \boldsymbol{T}_\theta = r\sin\phi\,\langle r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi\rangle$$

So $\mathrm{d}S = r^2\sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta$. This gives

$$\int_0^{2\pi}\int_0^{\pi} 3\left(r\sin\phi\cos\theta\right)^2 r^2\sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta = 4\pi r^4.$$

2. Because of the limits in x and y, you should use a graph parametrization. You get

$$\int_0^1\int_0^1 240xy\sqrt{1+4x^2+4y^2}\,\mathrm{d}x\,\mathrm{d}y = 244 - 2\,(5)^{5/2}$$

3. The answer is something like

$$2\int_0^1\int_0^1 yz\,\mathrm{d}y\,\mathrm{d}z + \int_0^1\int_0^1 y\,\mathrm{d}y\,\mathrm{d}x + \int_0^1\int_0^1 z\,\mathrm{d}z\,\mathrm{d}x = \frac{1}{2}+\frac{1}{2}+\frac{1}{2} = \frac{3}{2}$$

4. The plane has normal $\boldsymbol{n} = \langle 2,3,6\rangle$. Then $\|\boldsymbol{n}\|\cos\theta = \boldsymbol{n}\cdot\boldsymbol{k} = 6$. Thus $(\cos\theta)^{-1} = \sqrt{4+9+36}/6 = 7/6$. To find the region D in the $x-y$ plane, set $z=0$ in the planar equation to get the line $2x+3y-2=0$. Thus we get

$$\int_0^1\int_0^{2(1-x)/3} 5\,\frac{7}{6}\,\mathrm{d}y\,\mathrm{d}x = \frac{35}{18}$$

13. (§7.6) Given some oriented surface, S, parametrized by $\mathbf{\Phi}$, and some vector field $\boldsymbol{F}$, find the surface integral of $\boldsymbol{F}$,

$$\iint_S \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S}$$

1. $\boldsymbol{F} = \left\langle x^2, y, z^3 \right\rangle$, S is the surface of the cube bounded by $x = 0, x = 2, y = \pm 1, z = \pm 1$. Use outward normals.
2. $\boldsymbol{F} = \langle 0, 0, \cos(xy + 2^z) \rangle$, S is the part of the cylinder $x^2 + y^2 = 1$ with $0 \le z \le 2$, using the usual parametrization and an outward normal.
3. $\boldsymbol{F} = \langle y + z, 2x + y, y \rangle$, S is the surface of the triangle with corners $(1, 0, 0)$, $(0, 2, 0)$, $(0, 0, 3)$ with normal pointing away from the origin.
4. $\boldsymbol{F} = \langle x, y, -z \rangle$, S is the surface of the ellipsoid $x^2 + \frac{1}{4}y^2 + \frac{1}{9}z^2 = 9$. Use the parametrization $\mathbf{\Phi}(\phi, \theta) = (3 \sin\phi \cos\theta, 6 \sin\phi \sin\theta, 9 \cos\phi)$.
5. $\boldsymbol{F} = \langle 2y, 2x, z \rangle$, S is the part of the cone $x^2 = y^2 + z^2$ inside the sphere $(x - 8)^2 + y^2 + z^2 = 49$. Assume $\boldsymbol{n} \cdot \boldsymbol{i} < 0$.
6. $\boldsymbol{F} = \left\langle 3, x^2, y \right\rangle$, S is the part of the paraboloid $z = x^2 + y^2$ above the xy plane and below the plane $x + y + z = 9$. Assume $\boldsymbol{n} \cdot \boldsymbol{k} > 0$.

Answer:

1. You have to integrate over each of the six surfaces of the cube. This can be rather boring. In each case the normal will be in one of the three principal directions, which simplifies your integral. I get

$$\int_0^2 \int_{-1}^1 \left\langle x^2, y, -1 \right\rangle \cdot \langle 0, 0, -1 \rangle \,\mathrm{d}y\,\mathrm{d}x + \int_0^2 \int_{-1}^1 \left\langle x^2, y, 1 \right\rangle \cdot \langle 0, 0, 1 \rangle \,\mathrm{d}y\,\mathrm{d}x +$$
$$\int_0^2 \int_{-1}^1 \left\langle x^2, -1, z^3 \right\rangle \cdot \langle 0, -1, 0 \rangle \,\mathrm{d}z\,\mathrm{d}x + \int_0^2 \int_{-1}^1 \left\langle x^2, 1, z^3 \right\rangle \cdot \langle 0, 1, 0 \rangle \,\mathrm{d}z\,\mathrm{d}x +$$
$$\int_{-1}^1 \int_{-1}^1 \left\langle 0, y, z^3 \right\rangle \cdot \langle -1, 0, 0 \rangle \,\mathrm{d}z\,\mathrm{d}y + \int_{-1}^1 \int_{-1}^1 \left\langle 4, y, z^3 \right\rangle \cdot \langle 1, 0, 0 \rangle \,\mathrm{d}z\,\mathrm{d}y$$

   The answer I get is 32.
2. The answer is 0; note the field is always orthogonal to the normal, so you are integrating 0, which is 0.
3. The integral is something like

$$\int_0^1 \int_0^{2-2x} y - 6x + 9 \,\mathrm{d}y\,\mathrm{d}x.$$

4. When we study the divergence theorem later, we will see this is the volume of the ellipsoid. I believe the answer is 216π.

5. This surface was considered in Problem 11. Use the parametrization

$$\mathbf{\Phi}(r,\theta) = \langle r, r\cos\theta, r\sin\theta\rangle, \quad \text{which gives} \quad \boldsymbol{T}_r \times \boldsymbol{T}_\theta = \langle r, -r\cos\theta, -r\sin\theta\rangle$$

This parametrization is orientation reversing because $\boldsymbol{T}_r \times \boldsymbol{T}_\theta \cdot \boldsymbol{i} > 0$. So we will use $\mathrm{d}\boldsymbol{S} = \boldsymbol{T}_\theta \times \boldsymbol{T}_r \,\mathrm{d}r\,\mathrm{d}\theta$, giving

$$\int_0^{2\pi}\int_{4-\sqrt{17/2}}^{4+\sqrt{17/2}} \langle 2r\cos\theta, 2r, r\sin\theta\rangle \cdot \langle -r, r\cos\theta, r\sin\theta\rangle \,\mathrm{d}r\,\mathrm{d}\theta = \frac{\pi}{3} r^3\Big|_{4-\sqrt{17/2}}^{4+\sqrt{17/2}} = \frac{113\pi}{3}\sqrt{\frac{17}{2}}$$

6. This is kind of difficult. I made some substitutions and got

$$\iint_D \langle 3, u^2, v\rangle \cdot \langle -2u, -2v, 1\rangle \,\mathrm{d}u\,\mathrm{d}v = \iint_D -6u - 2u^2v + v \,\mathrm{d}u\,\mathrm{d}v$$

$$= \iint_{\tilde{D}} -6\left(\tilde{u} - \frac{1}{2}\right) - 2\left(\tilde{u} - \frac{1}{2}\right)^2\left(\tilde{v} - \frac{1}{2}\right) + \left(\tilde{v} - \frac{1}{2}\right) \mathrm{d}\tilde{u}\,\mathrm{d}\tilde{v}$$

$$= \int_0^{2\pi}\int_0^{\sqrt{19/2}} \left[\frac{11}{4} - 6r\cos\theta - 2r^3\sin\theta\cos^2\theta + 2r^2\sin\theta\cos\theta\right] r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= \frac{11}{2}\frac{19}{4}\pi - \frac{4}{15}\left(\frac{19}{2}\right)^{5/2}$$

YMMV.

14. (misc.) Let C be a curve representing the intersection of the sphere $x^2 + y^2 + z^2 = r^2$ and the plane $x + y + z = 0$. Let $f(x,y,z) = x^2$ be the mass density of a wire running along C. What is the total mass of the wire.

Answer: This is essentially #13 from §7.1. The hard part is getting the parametrization. I suggest you try to find one of the form $\boldsymbol{c}(\theta) = r\cos\theta\boldsymbol{u} + r\sin\theta\boldsymbol{v}$, where $\boldsymbol{u}, \boldsymbol{v}$ are unit vectors which are orthogonal to one another, and which are parallel to the plane, *i.e.*, which are normal to $\langle 1,1,1\rangle$. If you choose $\boldsymbol{u} = \langle 0, 1/\sqrt{2}, -1/\sqrt{2}\rangle$, and $\boldsymbol{v} = \langle -2/\sqrt{6}, 1/\sqrt{6}, 1/\sqrt{6}\rangle$, you will find that the x coordinate is $-(2r\sin\theta)/\sqrt{6}$. We will also have

$$\boldsymbol{c}'(\theta) = -r\sin\theta\boldsymbol{u} + r\cos\theta\boldsymbol{v},$$

the length of which is r. Thus our path integral is

$$\int_0^{2\pi} \frac{4}{6} r^2 \sin^2\theta r \,\mathrm{d}\theta = \frac{2}{3}\pi r^3$$

15. (misc.) Let $\phi(x,y,z) = e^x + yz$ represent the electric potential in space. What is the work done by the field $-\nabla\phi$ on a particle which moves from $(0,2,3)$ to $(3,-2,1)$.

Answer: This is a line integral. The answer should be

$$-\phi(x,y,z)\Big|_{(0,2,3)}^{(3,-2,1)} = -e^3 + 2 + 1 + 6 = 9 - e^3$$