Final Exam Preparation (COMPREHENSIVE VERSION)

The Final Exam is Wednesday, March 16th, and is comprehensive, covering all the material we have looked at this quarter, but with a particular emphasis on chapters §8.1–8.4. *Please bring a blue book for the exam.* You may not use a calculator or notes. The following formulæ will be provided on your exam:

$$\nabla \quad \text{abbreviates} \quad \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z} \qquad \mathrm{d}\boldsymbol{s} = \boldsymbol{c}'(t)\,\mathrm{d}t$$

$$\mathrm{d}\boldsymbol{S} = \boldsymbol{T}_u \times \boldsymbol{T}_v\,\mathrm{d}u\,\mathrm{d}v = \left(\frac{\partial \boldsymbol{\Phi}}{\partial u} \times \frac{\partial \boldsymbol{\Phi}}{\partial v}\right)\mathrm{d}u\,\mathrm{d}v \qquad \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

Everything else must be committed to memory.

What follows are questions similar to your homework assignments. You should be prepared to answer at least questions like these. This list may not be exhaustive.

Exam 1 Prep Sheet:

1. (§1.1–1.3) What's the difference between a scalar and a vector?

2. (§1.1–1.3) Let $\boldsymbol{u} = \langle 1, 2, -3\rangle$, $\boldsymbol{v} = \langle -2, 4, 1\rangle$, $\boldsymbol{w} = \langle -3, 0, 1\rangle$.

1. Find $5\boldsymbol{u}$.
2. Find $\boldsymbol{u} \cdot \boldsymbol{u}$.
3. Find $\|\boldsymbol{u}\|$.
4. Find a unit length vector in the same direction as $\boldsymbol{u}$.
5. Find $\|\alpha \boldsymbol{u}\|$, where α is a real number.
6. Find $\boldsymbol{u} \cdot \boldsymbol{v}$.
7. Find the angle subtended by $\boldsymbol{u}$ and $\boldsymbol{v}$.
8. Find $\boldsymbol{v} \times \boldsymbol{w}$.
9. Find a vector perpendicular to both $\boldsymbol{v}$ and $\boldsymbol{w}$.
10. Find a unit length vector perpendicular to both $\boldsymbol{v}$ and $\boldsymbol{w}$.
11. Find $\boldsymbol{u} \times \boldsymbol{v}$.
12. Find $\boldsymbol{w} \cdot (\boldsymbol{u} \times \boldsymbol{v})$.
13. Find $\boldsymbol{u} \times \boldsymbol{u}$.

3. (§1.1–1.3) Let $\boldsymbol{u} = \langle -2, 1, 4\rangle$, and let P be the point $(0, 4, 1)$.

1. Find a path $\boldsymbol{l}(t)$ which traces out the line through P and parallel to $\boldsymbol{u}$.
2. Find the equation of the plane containing P with normal $\boldsymbol{u}$.
3. Find a path $\boldsymbol{m}(t)$ which traces out a line through P and perpendicular to $\boldsymbol{u}$. (*Hint:* There are many answers. One way to do this is pick some arbitrary vector $\boldsymbol{v}$ not parallel to $\boldsymbol{u}$, then find a vector perpendicular to both $\boldsymbol{u}$ and $\boldsymbol{v}$.)

4. (§1.4) Convert the following points, given in cylindrical coordinates, to cartesian coordinates, and to spherical coordinates:

$$P = (1, \pi/3, 1) \qquad Q = (2, \pi, 0) \qquad R = \left(1, \pi/6, -\sqrt{3}\right)$$

5. (§1.4) Convert the following points, given in spherical coordinates, to cartesian coordinates, and to cylindrical coordinates:

$$P = [(\rho, \theta, \phi) = (1, \pi/2, \pi/3)] \qquad Q = [(\rho, \theta, \phi) = (2, 3\pi/2, \pi/2)]$$

6. (§2.1) Define a level set for a function $f\colon \mathbb{R}^n \to \mathbb{R}$

7. (§2.1) Describe the level sets of the following functions. Where appropriate, graph a few of the level sets.

1. $f(x, y) = x^2 + y^2$.
2. $g(x, y) = 2x + y$.
3. $h(x, y) = xy$.
4. $l(x, y, z) = x - y + 2z$.
5. $m(x, y, z) = x^2 + y^2$.
6. $n(x, y, z) = x^2 + y^2 + z^2$.
7. $p(x, y, z) = x^2 + y^2 - z$.

8. (§2.1) Sketch the graphs of the following functions:

1. $f(x, y) = x^2 + y^2$.
2. $g(x, y) = 2x + y$.

9. (§2.3) Find ∇f for

1. $f(x, y) = x^2 + y^2$.
2. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$.
3. $f(x, y) = e^{x+y}$.
4. $f(x, y, z) = e^{x+y}\cos z + y^2$.

10. (§2.3) Find the equation of the tangent plane to the graph of f at the point P for

1. $f(x, y) = x^2 + y^2$, $P = (3, 4, 25)$.
2. $f(x, y) = e^{x+y}$, $P = \big(2, 1, e^3\big)$.

11. (§2.3) In your own words, what does it mean for a function $f\colon \mathbb{R}^n \to \mathbb{R}$ to be "differentiable" at a point x_0? If f is differentiable at x_0, does f have to be smooth at this point? Is there a unique tangent plane at this point?

12. (§2.3) Remember the theorem: if all the partial derivatives of $f\colon \mathbb{R}^n \to \mathbb{R}$ exist and are continuous at x_0, then f is differentiable at x_0.

13. (§2.4) What kind of curve does $\boldsymbol{c}(t)$ trace out for

1. $\boldsymbol{c}(t) = \langle 1 + 2t, 3, 4 - t\rangle$
2. $\boldsymbol{c}(t) = \langle \cos 2t, \sin 2t\rangle$.
3. $\boldsymbol{c}(t) = \langle \cos t, \sin t, 3t\rangle$.

14. (§2.4) Given the path $\boldsymbol{c}(t)$, find its velocity function $\boldsymbol{c}'(t)$, and its speed function $\|\boldsymbol{c}'(t)\|$

1. $\boldsymbol{c}(t) = \langle 1 + 3t, 3, 1 - 4t\rangle$
2. $\boldsymbol{c}(t) = \langle \cos t, \sin t, 3t\rangle$.
3. $\boldsymbol{c}(t) = \big\langle t^2, t^3, t\big\rangle$.
4. $\boldsymbol{c}(t) = \big\langle t, t^3 + 1/(4t)\big\rangle$.
5. $\boldsymbol{c}(t) = \big\langle t/\sqrt{2}, t/\sqrt{2}, \ln(\sec t)\big\rangle$.

15. (§2.4) Given the path $\boldsymbol{c}(t)$, and a time t_0, find the tangent vector to $\boldsymbol{c}(t_0)$ for

1. $\boldsymbol{c}(t) = \langle 1+3t, 3, 1-4t \rangle$, $t_0 = 1$.
2. $\boldsymbol{c}(t) = \left\langle t^2, -3t, 1+t^3 \right\rangle$, $t_0 = 2$.

16. (§2.5) Given the path $\boldsymbol{c}(t)$ in $\mathbb{R}^n$ and a real valued function $f(\boldsymbol{x})$ on n variables, find $\frac{\mathrm{d}}{\mathrm{d}t}\left[f \circ \boldsymbol{c}(t)\right]$

1. $\boldsymbol{c}(t) = \langle 1+3t, 3, 1-4t \rangle$, $f(x,y,z) = xy - yz$.
2. $\boldsymbol{c}(t) = \langle \cos t, \sin t, t \rangle$, $f(x,y,z) = x^2 + y^2 - z$
3. $\boldsymbol{c}(t) = \left\langle t, t^2, t^3 \right\rangle$, $f(x,y,z) = xyz + e^{xy}$.

17. (§2.6) Given the real valued function $f(\boldsymbol{x})\colon \mathbb{R}^n \to \mathbb{R}$, and the vector $\boldsymbol{v}$, find the directional derivative of f along $\boldsymbol{v}$ at a given point, for

1. $f(x,y) = x^2 + y^2$, $\boldsymbol{v} = \langle 1, 3 \rangle$, at the point $(3,4)$.
2. $f(x,y,z) = \sqrt{x^2+y^2+z^2}$, $\boldsymbol{v} = \langle -1, 0.5, 3 \rangle$, at the point $(1,0,-1)$.
3. $f(x,y) = e^{x+y}$, $\boldsymbol{v} = \left\langle \sqrt{2}/2, \sqrt{2}/2 \right\rangle$, at the point $(1,-1)$.
4. $f(x,y,z) = e^{x+y}\cos z + y^2$, $\boldsymbol{v} = \left\langle 1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3} \right\rangle$, at the point $(2,1,\pi/2)$.
5. $f(x,y,z) = x + y\ln z$, $\boldsymbol{v} = \langle 1/3, 1, -1 \rangle$ at the point $(1,2,0)$.

18. (§2.6) Given the real valued function $f(\boldsymbol{x})\colon \mathbb{R}^n \to \mathbb{R}$, and a point in $\mathbb{R}^n$, find the direction (*i.e.*, unit vector) of maximal increase of f at that point for

1. $f(x,y) = x^2 + y^2$, at the point $(3,4)$.
2. $f(x,y) = xy + 1/(xy)$, at the point $(1,2)$.
3. $f(x,y,z) = xyz$, at the point $(1,0,2)$.
4. $f(x,y,z) = 2x + 3y - 4z$, at the point $(1,-1,5)$.
5. $f(x,y,z) = x^2 + y^2 - z$, at the point $(0,1,1)$.

19. (§3.2) Find the linear Taylor formula (*i.e.*, linearization) of $f(x,y) = x^2 + y^2$ at the point $(1,2)$.

20. (§3.2) Find the quadratic Taylor formula of $f(x,y) = x^2 + y^2$ at the point $(1,2)$.

21. (§4.2) Given path $\boldsymbol{c}(t)$, find the arc length of the path for t between t_0 and t_1, for

1. $\boldsymbol{c}(t) = \langle 1+3t, 3, 1-4t \rangle$, $t_0 = 0$, $t_1 = 3$.
2. $\boldsymbol{c}(t) = \langle \cos t, \sin t, 3t \rangle$, $t_0 = 0$, $t_1 = 4\pi$.
3. $\boldsymbol{c}(t) = \left\langle t/\sqrt{2}, t/\sqrt{2}, \ln(\sec t) \right\rangle$, $t_0 = 0$, $t_1 = \pi/3$.
4. $\boldsymbol{c}(t) = \left\langle t, t^3/3 + 1/(4t) \right\rangle$, $t_0 = 1$, $t_1 = 2$.

22. (§4.4) Match four of the following vector fields to their graphical representation in Figure 1. (Two of the following fields are not plotted.)

1. $\boldsymbol{F}(x,y) = \langle y, x \rangle$.
2. $\boldsymbol{F}(x,y) = \left\langle x^2, y^2 \right\rangle$.
3. $\boldsymbol{F}(x,y) = \langle 2, 5 \rangle$.
4. $\boldsymbol{F}(x,y) = \langle y, -x \rangle$.
5. $\boldsymbol{F}(x,y) = \langle 2x, 2y \rangle$.
6. $\boldsymbol{F}(x,y) = \langle -y, x \rangle$.

23. (§4.4) Draw some flow lines for the vector fields in Figure 1.

24. (§4.4) Which of the vector fields plotted in Figure 1 might be gradient fields, that is of the form $\nabla\phi$ for some scalar field ϕ? For those that are, try to draw some level sets for the function ϕ over the vector field.

25. (§4.4) Given the vector field $\boldsymbol{F}(x, y, z)$, find div $\boldsymbol{F} = \nabla \cdot \boldsymbol{F}$, and **curl** $\boldsymbol{F} = \nabla \times \boldsymbol{F}$ for

1. $\boldsymbol{F}(x, y, z) = \left\langle xy + z, x^2 - y, z^2 - 3 \right\rangle$.
2. $\boldsymbol{F}(x, y, z) = \langle yz, xz, xy \rangle$.
3. $\boldsymbol{F}(x, y, z) = \langle 2x, 2y, -1 \rangle$.
4. $\boldsymbol{F}(x, y, z) = \langle y, -x, 0 \rangle$.
5. $\boldsymbol{F} = \nabla\phi$, where $\phi(x, y, z) = 2xy^2 - ze^y$.
6. $\boldsymbol{F} = \nabla \times \boldsymbol{G}$, where $\boldsymbol{G}(x, y, z) = \left\langle xy + z, x^2 - y, z^2 - 3 \right\rangle$.

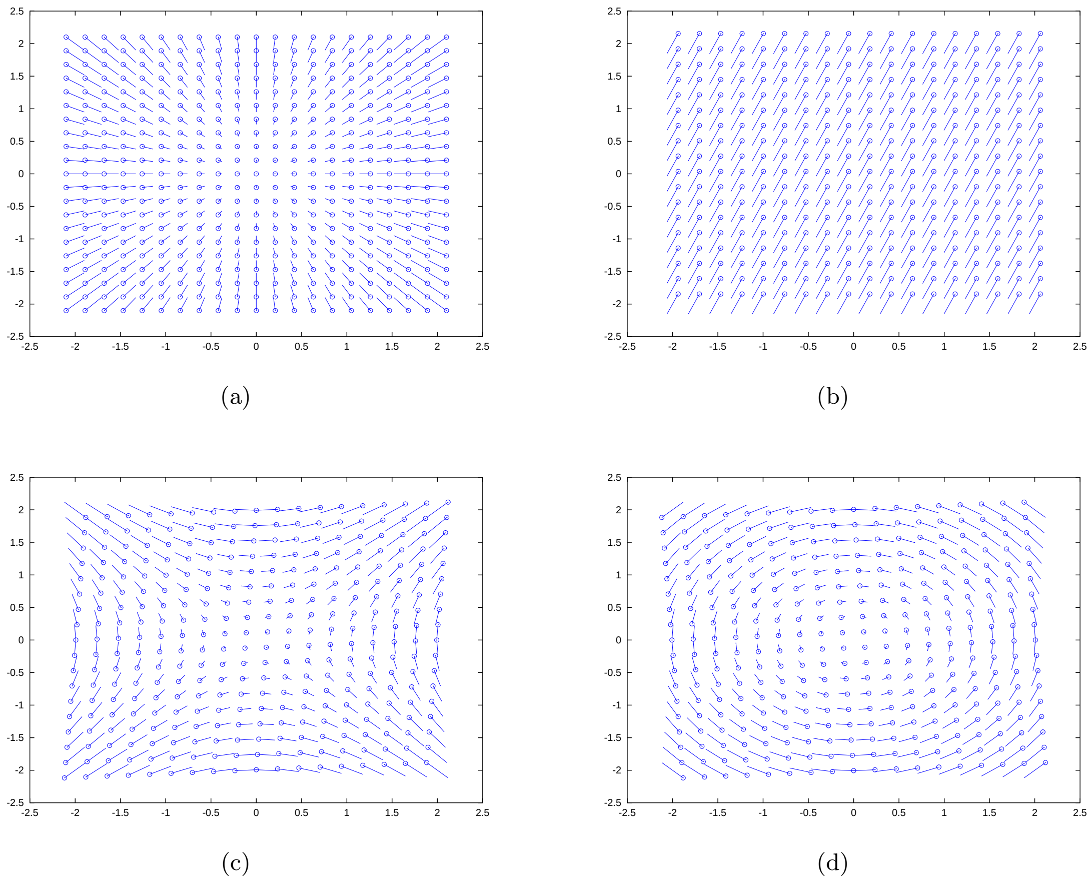

Figure 1: Four vector fields. Circles represent the tips of the vectors (proper arrowheads were not available).

Exam 2 Prep Sheet:

26. (§5.1–5.4) Given function $f(x,y)$ and region $D \subseteq \mathbb{R}^2$, evaluate

$$\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

1. $f(x,y) = 2xy$, D is the region bounded by $y = 0, x = 2, y = x^2$.
2. $f(x,y) = y^2 \cos x$, D is region bounded by $x = y^3, y = -1, y = 1, x = 3$.

27. (§5.1–5.4) Given function $f(x,y,z)$ and region $D \subseteq \mathbb{R}^3$, evaluate

$$\iiint_D f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

1. $f(x,y,z) = \sqrt{x^2+z^2}$, D is bounded by $y = x^2 + z^2, y = 4$.
2. $f(x,y,z) = x + y + z$, D is bounded by $x = y^2, x = z, z = 0, x = 1$.

28. (§6.1) Given a change of variables transformation $T \colon D^* \subseteq \mathbb{R}^n \to \mathbb{R}^n$, and the domain, D^*, find the image of the transformation, $D = T(D^*)$. Is the transformation one-to-one?

1. $(x,y) = T(r,\theta) = (r\cos\theta, r\sin\theta)$, and $D^* = [0,4] \times [0,\pi/2]$.
2. $(x,y,z) = T(r,\theta,z) = (r\cos\theta, r\sin\theta, z)$, and $D^* = [0,1] \times [0,\pi] \times [0,1]$.

29. (§6.1) Given a change of variables transformation $T \colon D^* \subseteq \mathbb{R}^n \to \mathbb{R}^n$, and a set D, find a domain, D^*, such that $D = T(D^*)$. Is the transformation one-to-one?

1. $(x,y) = T(u,v) = (u/v, v)$, and D is the region bounded by the curves $xy = 1, xy = 4, y = 1, y = 3$.

30. (§6.2) Given a change of variables transformation $T \colon D^* \subseteq \mathbb{R}^n \to \mathbb{R}^n$, find the Jacobian Determinant of the transformation.

1. With $(x,y) = T(r,\theta) = (r\cos\theta, r\sin\theta)$, find $\frac{\partial(x,y)}{\partial(r,\theta)}$.
2. With $(x,y) = T(u,v) = (u/v, v)$, find $\frac{\partial(x,y)}{\partial(u,v)}$.
3. With $(x,y) = T(u,v) = ((u+v)/2, (u-v)/2)$, find $\frac{\partial(x,y)}{\partial(u,v)}$.
4. With $(x,y) = T(u,v) = (au + bv + c, du + fv + g)$, find $\frac{\partial(x,y)}{\partial(u,v)}$.
5. With $(x,y,z) = T(u,v,w) = (au, bv, cw)$, find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$.
6. With $(x,y,z) = T(r,\theta,z) = (r\cos\theta, r\sin\theta, z)$, find $\frac{\partial(x,y,z)}{\partial(r,\theta,z)}$.
7. With $(x,y,z) = T(\rho,\theta,\phi) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$, find $\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}$

31. (§6.2) Given a change of variables transformation $T \colon D^* \subseteq \mathbb{R}^n \to \mathbb{R}^n$, some set D, and a function $f(x,y)$, evaluate

$$\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y \quad \text{as an integral of the form} \quad \iint_{D^*} f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\,\mathrm{d}v$$

You may be forced to find D^* yourself.

1. $f(x,y) = \sin\left(x^2 + 2xy + y^2\right)$, D is the triangle with corners $(0,0)$,$(1,0)$,$(0,1)$ using the change of variables $x = (u+v)/2, y = (u-v)/2$.

2. $f(x,y) = e^{xy}$, D is bounded by $xy = 1, xy = 4, y = 1, y = 3$, and using the change of variables $x = u/v, y = v$.

32. (§7.1) Given the scalar field $f(x,y,z)$ and path $\boldsymbol{c}(t)\colon [a,b] \to \mathbb{R}^3$ evaluate the path integral

$$\int_{\boldsymbol{c}} f \, \mathrm{d}s$$

1. $f(x,y,z) = x^2 + y^2$, along $\boldsymbol{c}(t) = \langle r\cos t, r\sin t, t\rangle$ for $a \le t \le b$.
2. $f(x,y,z) = z$, along $\boldsymbol{c}(t) = \langle \cos t, \sin t, t\rangle$ for $0 \le t \le 2\pi$.
3. $f(x,y,z) = \cos\left(\sqrt{z} + \sqrt{y}\right)$, along $\boldsymbol{c}(t) = \left\langle 1, t^2/4, t^2/4\right\rangle$ for $0 \le t \le 1$.

33. (§7.2) Given the vector field $\boldsymbol{F}(x,y,z)$ and path $\boldsymbol{c}(t)\colon [a,b] \to \mathbb{R}^3$ evaluate the line integral

$$\int_{\boldsymbol{c}} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s}$$

1. $\boldsymbol{F} = \left\langle 3x + 4y, 2x + 3y^2\right\rangle$, along $\boldsymbol{c}(t) = \langle 2\cos t, 2\sin t\rangle$, $0 \le t \le 2\pi$.
2. $\boldsymbol{F} = \left\langle 2xy, x^2 + z, y\right\rangle$, along the path which traces a straight line from $(1,0,2)$ to $(3,4,1)$.
3. $\boldsymbol{F} = \langle yz, xz, xy\rangle$ along some path $\boldsymbol{c}$ with $\boldsymbol{c}(a) = (-1,-1,2)$, and $\boldsymbol{c}(b) = (-2,1,-1)$.
4. $\boldsymbol{F} = \left\langle 2xz + y^2z^2, 2xyz^2, x^2 + 2xy^2z\right\rangle$ along the path $\boldsymbol{c}(t) = \left\langle \cos(\pi t) + e^t, -\sin(\pi t) + \log(1+t), t^{12}\right\rangle$ for $0 \le t \le 1$.

34. (§7.2) Given a vector field $\boldsymbol{F}$, determine if it is conservative. If it is conservative, find a ϕ such that $\boldsymbol{F} = \nabla\phi$.

1. $\boldsymbol{F} = \left\langle 2xy, x^2 + z, y\right\rangle$.
2. $\boldsymbol{F} = \langle xy, z, x\rangle$.
3. $\boldsymbol{F} = \left\langle x^2y + 1, \frac{1}{3}x^3 + 1, y\right\rangle$.
4. $\boldsymbol{F} = \frac{1}{x^2+y^2+z^2}\langle x, y, z\rangle$.
5. $\boldsymbol{F} = \langle -\sin x\cos y, \cos x\cos y, 1\rangle$.
6. $\boldsymbol{F} = \frac{1}{(x^2+y^2+z^2)^{3/2}}\langle x, y, z\rangle$.

35. (§7.3) Given a parametrization of a surface, $\boldsymbol{\Phi}(u,v)$, find $\boldsymbol{T}_u \times \boldsymbol{T}_v$. Is the parametrization regular?

1. $\boldsymbol{\Phi}(u,v) = \left\langle v\cos u, v\sin u, v^2\right\rangle$.
2. $\boldsymbol{\Phi}(u,v) = \langle (1+\cos u)\cos v, (1+\cos u)\sin v, \sin v\rangle$.
3. $\boldsymbol{\Phi}(u,v) = \langle u, v, g(u,v)\rangle$.
4. $\boldsymbol{\Phi}(u,v) = \left\langle u, v, u^2 + v^{3/2}\right\rangle$.
5. $\boldsymbol{\Phi}(u,v) = \left\langle u^2, v - u, vu\right\rangle$.

36. (§7.4) Given some surface, S, find a regular parametrization $\boldsymbol{\Phi}(u,v)$, and domain $D \subseteq \mathbb{R}^2$ such that $S = \boldsymbol{\Phi}(D)$. Find $\mathrm{d}\boldsymbol{S}$ and $\mathrm{d}S = \|\mathrm{d}\boldsymbol{S}\|$. Set up an integral for the surface area of S.

1. S is the part of the cone $x^2 = y^2 + z^2$ inside the sphere $(x-8)^2 + y^2 + z^2 = 49$.
2. S is the part of the paraboloid $z = x^2 + y^2$ above the xy plane and below the plane $x + y + z = 9$.

37. (§7.5) Given some oriented surface, S, parametrized by $\mathbf{\Phi}$, and some scalar field f, find the integral of f over S:

$$\iint_S f \, \mathrm{d}S$$

1. $f(x, y, z) = 3x^2$ over the sphere of radius r.
2. $f(x, y, z) = 240xy$ over S, which is the paraboloid $z = x^2 + y^2$ for $x \in [0, 1]$, $y \in [0, 1]$.
3. $f(x, y, z) = yz$ over S, which is the boundary of the cube $[0, 1] \times [0, 1] \times [0, 1]$. That is, the cube with corners $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 1, 0)$, $(0, 1, 1)$, $(1, 0, 1)$, $(1, 1, 1)$.
4. $f(x, y, z) = 5$ over S, which is the plane $2x + 3y + 6z - 2 = 0$, with $x \geq 0$, $y \geq 0$, $z \geq 0$.

38. (§7.6) Given some oriented surface, S, parametrized by $\mathbf{\Phi}$, and some vector field $\boldsymbol{F}$, find the surface integral of $\boldsymbol{F}$,

$$\iint_S \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S}$$

1. $\boldsymbol{F} = \left\langle x^2, y, z^3 \right\rangle$, S is the surface of the cube bounded by $x = 0, x = 2, y = \pm 1, z = \pm 1$. Use outward normals.
2. $\boldsymbol{F} = \langle 0, 0, \cos(xy + 2^z) \rangle$, S is the part of the cylinder $x^2 + y^2 = 1$ with $0 \leq z \leq 2$, using the usual parametrization and an outward normal.
3. $\boldsymbol{F} = \langle y + z, 2x + y, y \rangle$, S is the surface of the triangle with corners $(1, 0, 0)$, $(0, 2, 0)$, $(0, 0, 3)$ with normal pointing away from the origin.
4. $\boldsymbol{F} = \langle x, y, -z \rangle$, S is the surface of the ellipsoid $x^2 + \frac{1}{4}y^2 + \frac{1}{9}z^2 = 9$. Use the parametrization $\mathbf{\Phi}(\phi, \theta) = (3 \sin\phi \cos\theta, 6 \sin\phi \sin\theta, 9 \cos\phi)$.
5. $\boldsymbol{F} = \langle 2y, 2x, z \rangle$, S is the part of the cone $x^2 = y^2 + z^2$ inside the sphere $(x-8)^2 + y^2 + z^2 = 49$. Assume $\boldsymbol{n} \cdot \boldsymbol{i} < 0$.
6. $\boldsymbol{F} = \left\langle 3, x^2, y \right\rangle$, S is the part of the paraboloid $z = x^2 + y^2$ above the xy plane and below the plane $x + y + z = 9$. Assume $\boldsymbol{n} \cdot \boldsymbol{k} > 0$.

39. (misc.) Let C be a curve representing the intersection of the sphere $x^2 + y^2 + z^2 = r^2$ and the plane $x + y + z = 0$. Let $f(x, y, z) = x^2$ be the mass density of a wire running along C. What is the total mass of the wire.

40. (misc.) Let $\phi(x, y, z) = e^x + yz$ represent the electric potential in space. What is the work done by the field $-\nabla\phi$ on a particle which moves from $(0, 2, 3)$ to $(3, -2, 1)$.

Final Exam Prep Sheet:

41. (§8.1) State Green's Theorem.

42. (§8.1) Given some region in the plane, D, bounded by the closed curve, C, find the area of D.

1. C is traced out by $\boldsymbol{c}(t) = \left\langle \sin t \cos t, \sin^2 t, 0 \right\rangle$ for $0 \leq t \leq \pi$. (From last year's final)

2. C is traced out by $r = r(\theta)$ for $0 \le \theta \le 2\pi$. That is, the curve C is given by $\boldsymbol{c}(t) = \langle r(t)\cos t, r(t)\sin t, 0\rangle$ for $0 \le t \le 2\pi$. (Note that in order for C to be a closed curve we need to have $r(0) = r(2\pi)$, and $r(t) \ge 0$)
3. C is traced out by $r = r(\theta) = 2\pi\theta - \theta^2$ for $0 \le \theta \le 2\pi$.
4. Let C be traced out by $\boldsymbol{c}(t) = \left\langle \cos^2 t, \cos t \sin t, 0 \right\rangle$, for $-\pi/2 \le t \le \pi/2$. (Oddly enough, this problem doesn't really require any calculus.)
5. Let C be traced out by $\boldsymbol{c}(t) = \left\langle \cos^3 t, \cos^2 t \sin t, 0 \right\rangle$, for $-\pi/2 \le t \le \pi/2$.
6. Let C be one loop of the lemniscate parametrized by $\boldsymbol{R}(t) = \left\langle \sqrt{\cos 2t}\cos t, \sqrt{\cos 2t}\sin t, 0 \right\rangle$ for $-\pi/4 \le t \le \pi/4$.

43. (§8.2) State Stokes' Theorem.

44. (§8.2) Given some surface, S, and a vector field $\boldsymbol{F}$ find the surface integral of the curl:

$$\iint_S (\nabla \times \boldsymbol{F}) \cdot \mathrm{d}\boldsymbol{S}.$$

1. $\boldsymbol{F} = \langle -z - y, x + y, z + x \rangle$, and S is the hemisphere $x^2 + y^2 + z^2 = 1$, with $z \ge 0$, oriented such that $\boldsymbol{n} \cdot \boldsymbol{k} \ge 0$. (From last year's final)
2. $\boldsymbol{F} = \left\langle x^2 + y - 4, 3xy, 2xz + z^2 \right\rangle$, S is the surface of the hemisphere $x^2+y^2+z^2 = 16$ above the xy plane.
3. $\boldsymbol{F} = \left\langle x \cos y e^z, x^2 - y^2, \arctan z + y^x \right\rangle$, and S is the (closed) surface given by $16x^2 + 9y^2 + 4z^2 = 1$.
4. $\boldsymbol{F} = \left\langle y^2, z^2, x^2 \right\rangle$, and S is the hemisphere $x^2 + y^2 + z^2 = 9$, with $z \ge 0$ oriented such that $\boldsymbol{n} \cdot \boldsymbol{k} \ge 0$.
5. $\boldsymbol{F} = \langle -y, x, 2x + z \rangle$, and S is the surface $\left\{ (x, y, z) \,\middle|\, z^2 = x^2 + y^2,\ 2 \le z \le 3 \right\}$ oriented such that $\boldsymbol{n} \cdot \boldsymbol{k} \le 0$.

45. (§8.2) Given some closed curve C parametrized by $\boldsymbol{c}(t)$, and a vector field $\boldsymbol{F}$, find the circulation of the vector field:

$$\int_{\boldsymbol{c}} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s}.$$

1. $\boldsymbol{F} = \boldsymbol{r} = \langle x, y, z \rangle$, with $\boldsymbol{c}(t) = \langle 2\cos t, 2\sin t, 0 \rangle$, for $0 \le t \le 2\pi$.
2. $\boldsymbol{F} = \langle 6xz, 2xz, 0 \rangle$, with C the boundary of triangle going from $(1, 0, 0)$ to $(0, 3, 0)$ to $(0, 0, 2)$.

46. (§8.3) Define what it means for a field, $\boldsymbol{F}$, defined over all of $\mathbb{R}^3$, to be conservative. Give some equivalent conditions.

47. (§8.3) Given a field, $\boldsymbol{F}$, determine if $\boldsymbol{F}$ is conservative, and if it is, find a potential for it, *i.e.*, find a ϕ such that $\boldsymbol{F} = \nabla\phi$.

1. $\boldsymbol{F} = \left\langle \cos x + 2yx, x^2, e^z \right\rangle$.
2. $\boldsymbol{F} = \langle xyz, xyz, xyz \rangle$.
3. $\boldsymbol{F} = \langle -y, x, 0 \rangle$.

48. (§8.3) Given a field, $\boldsymbol{F}$, and a curve, C, find the line integral

$$\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S}$$

1. $\boldsymbol{F} = \left\langle 5x^4 - 2xy^3, -3x^2y^2, 2z \right\rangle$, and C is the straight line from $(0,0,0)$ to $(2,-2,1)$.

49. (§8.4) State the Divergence Theorem.

50. (§8.4) Given some surface, S, and a vector field $\boldsymbol{F}$ find the integral of the flux:

$$\iint_S \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S}.$$

1. $\boldsymbol{F} = \left\langle \frac{y}{b^2}, \frac{-x}{a^2}, 1 \right\rangle$, S is the ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$. Use outward pointing normals.
2. $\boldsymbol{F} = \boldsymbol{r} = \langle x, y, z \rangle$, S is the surface of some set W with volume V. Use outward pointing normals.
3. $\boldsymbol{F} = \langle x + z, x + y, \cos(x) - z \rangle$, S is the surface of some set W with volume V. Use outward pointing normals.
4. $\boldsymbol{F} = \left\langle x^2, y^2, z^3 \right\rangle$, S is the surface of the cube bounded by $x = \pm 1, y = \pm 1, z = \pm 1$. Use outward pointing normals.
5. $\boldsymbol{F} = \left\langle x^2, y, z^3 \right\rangle$, S is the surface of the cube bounded by $x = 0, x = 2, y = \pm 1, z = \pm 1$. Use outward normals.
6. $\boldsymbol{F} = \left\langle 4x, -2y^2, z^2 \right\rangle$, S is the cylinder $\left\{ (x,y,z) \,\middle|\, x^2 + y^2 = 4,\ -3 \le z \le 3 \right\}$. Assume the normal points away from the z axis.
7. $\boldsymbol{F} = \left\langle \frac{1}{4}x, z, x - \frac{y}{2} \right\rangle$, and S is the boundary of the region W bounded by the inequalities $\frac{1}{16}x^2 + \frac{1}{4}y^2 + z^2 \le 1$, and $z \ge 0$. Assume outward normals. (Note in this case we are taking S to be the closed surface which is the boundary of W.)

51. (misc.) Let ϕ be a differentiable scalar field, let $\boldsymbol{F} = \nabla\phi / \|\nabla\phi\|$ be a vector field, and let W be the region in $\mathbb{R}^3$ defined by $\{(x,y,z) \mid \phi(x,y,z) \le 17\}$. Find $\iiint_W \nabla \cdot \boldsymbol{F}\, \mathrm{d}V$.

52. (misc.) Let $\boldsymbol{c}(t) = \left(e^{2t}/\sqrt{3}\right)\langle 1, 1, 1 \rangle$ be the path of a particle. Show that the particle is always moving in the direction of maximal increase of the scalar function $\phi(x,y,z) = x^2 + y^2 + z^2$.