Final Exam Preparation (COMPREHENSIVE VERSION)

The Final Exam is Wednesday, March 16th, and is comprehensive, covering all the material we have looked at this quarter, but with a particular emphasis on chapters §8.1–8.4. *Please bring a blue book for the exam.* You may not use a calculator or notes. The following formulæ will be provided on your exam:

$$\nabla \quad \text{abbreviates} \quad \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z} \qquad \mathrm{d}\boldsymbol{s} = \boldsymbol{c}'(t)\,\mathrm{d}t$$

$$\mathrm{d}\boldsymbol{S} = \boldsymbol{T}_u \times \boldsymbol{T}_v \,\mathrm{d}u\,\mathrm{d}v = \left(\frac{\partial \boldsymbol{\Phi}}{\partial u} \times \frac{\partial \boldsymbol{\Phi}}{\partial v}\right)\mathrm{d}u\,\mathrm{d}v \qquad \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

Everything else must be committed to memory.

What follows are questions similar to your homework assignments. You should be prepared to answer at least questions like these. This list may not be exhaustive.

Exam 1 Prep Sheet:

1. (§1.1–1.3) What's the difference between a scalar and a vector?

Answer: A vector comprises a direction and a scalar magnitude. A scalar can be conceived of as a vector in 1 dimension, and thus has trivial direction (*i.e.*, in the positive direction).

2. (§1.1–1.3) Let $\boldsymbol{u} = \langle 1, 2, -3 \rangle$, $\boldsymbol{v} = \langle -2, 4, 1 \rangle$, $\boldsymbol{w} = \langle -3, 0, 1 \rangle$.

1. Find $5\boldsymbol{u}$.
2. Find $\boldsymbol{u} \cdot \boldsymbol{u}$.
3. Find $\|\boldsymbol{u}\|$.
4. Find a unit length vector in the same direction as $\boldsymbol{u}$.
5. Find $\|\alpha\boldsymbol{u}\|$, where α is a real number.
6. Find $\boldsymbol{u} \cdot \boldsymbol{v}$.
7. Find the angle subtended by $\boldsymbol{u}$ and $\boldsymbol{v}$.
8. Find $\boldsymbol{v} \times \boldsymbol{w}$.
9. Find a vector perpendicular to both $\boldsymbol{v}$ and $\boldsymbol{w}$.
10. Find a unit length vector perpendicular to both $\boldsymbol{v}$ and $\boldsymbol{w}$.
11. Find $\boldsymbol{u} \times \boldsymbol{v}$.
12. Find $\boldsymbol{w} \cdot (\boldsymbol{u} \times \boldsymbol{v})$.
13. Find $\boldsymbol{u} \times \boldsymbol{u}$.

Answer:
1. $\langle 5, 10, -15 \rangle$
2. 14
3. $\sqrt{14}$
4. $(1/\sqrt{14})\boldsymbol{u} = \left\langle 1/\sqrt{14}, 2/\sqrt{14}, -3/\sqrt{14} \right\rangle$.
5. $|\alpha|\sqrt{14}$
6. $-2 + 8 - 3 = 3$
7. $\arccos\left(\boldsymbol{u} \cdot \boldsymbol{v}/\|\boldsymbol{u}\|\,\|\boldsymbol{v}\|\right) = \arccos\left(3/\sqrt{14}\sqrt{21}\right) \approx 80°$
8. $\boldsymbol{v} \times \boldsymbol{w} = \langle 4, -1, 12 \rangle$

9. $\boldsymbol{v} \times \boldsymbol{w}$ is perpendicular to both $\boldsymbol{v}$ and $\boldsymbol{w}$.

10. $\boldsymbol{v} \times \boldsymbol{w} / \|\boldsymbol{v} \times \boldsymbol{w}\| = \langle 4, -1, 12\rangle / \sqrt{161} = \left\langle 4/\sqrt{161}, -1/\sqrt{161}, 12/\sqrt{161}\right\rangle$.

3. (§1.1–1.3) Let $\boldsymbol{u} = \langle -2, 1, 4\rangle$, and let P be the point $(0, 4, 1)$.

1. Find a path $\boldsymbol{l}(t)$ which traces out the line through P and parallel to $\boldsymbol{u}$.
2. Find the equation of the plane containing P with normal $\boldsymbol{u}$.
3. Find a path $\boldsymbol{m}(t)$ which traces out a line through P and perpendicular to $\boldsymbol{u}$. (*Hint:* There are many answers. One way to do this is pick some arbitrary vector $\boldsymbol{v}$ not parallel to $\boldsymbol{u}$, then find a vector perpendicular to both $\boldsymbol{u}$ and $\boldsymbol{v}$.)

Answer: 1. One answer is $\boldsymbol{l}(t) = \langle 0, 4, 1\rangle + t\langle -2, 1, 4\rangle = \langle -2t, 4+t, 1+4t\rangle$.

2. Planes are of the form $ax+by+cz+d = 0$, where $\langle a, b, c\rangle$ (and any scalar multiple of it) are normal to the plane. So the answer should be of the form $-2x+y+4z+d = 0$. Plug in P as (x, y, z) to find the value of d. You should get $-2x + y + 4z - 8 = 0$.
3. First I will let $\boldsymbol{v} = \langle 0, 0, 1\rangle$, just for simplicity. Then I compute $\boldsymbol{w} = \boldsymbol{u} \times \boldsymbol{v} = \langle 1, 2, 0\rangle$, which is perpendicular to both $\boldsymbol{u}$ and $\boldsymbol{v}$. I only care that it is perpendicular to $\boldsymbol{u}$. If I had picked the "wrong" $\boldsymbol{v}$, this cross product would have been $\mathbf{0}$, and I would have known something was wrong and picked a different $\boldsymbol{v}$. Then I do as in the first part, getting $\boldsymbol{m}(t) = \langle 0, 4, 1\rangle + t\langle 1, 2, 0\rangle$.

4. (§1.4) Convert the following points, given in cylindrical coordinates, to cartesian coordinates, and to spherical coordinates:

$$P = (1, \pi/3, 1) \qquad Q = (2, \pi, 0) \qquad R = \left(1, \pi/6, -\sqrt{3}\right)$$

Answer:

$$P = \left(1/2, \sqrt{3}/2, 1\right) \qquad Q = (-2, 0, 0) \qquad R = \left(\sqrt{3}/2, 1/2, -\sqrt{3}\right)$$

5. (§1.4) Convert the following points, given in spherical coordinates, to cartesian coordinates, and to cylindrical coordinates:

$$P = [(\rho, \theta, \phi) = (1, \pi/2, \pi/3)] \qquad Q = [(\rho, \theta, \phi) = (2, 3\pi/2, \pi/2)]$$

Answer:

$$P = \left(0, \sqrt{3}/2, 1/2\right) \qquad Q = (0, -2, 0)$$

6. (§2.1) Define a level set for a function $f\colon \mathbb{R}^n \to \mathbb{R}$

Answer: A level set is a set of the form $\{\boldsymbol{x} \in \mathbb{R}^n \mid f(\boldsymbol{x}) = k\}$, that is, all points in $\mathbb{R}^n$ where f takes the value k. There is a different level set for each real constant k.

7. (§2.1) Describe the level sets of the following functions. Where appropriate, graph a few of the level sets.

1. $f(x, y) = x^2 + y^2$.
2. $g(x, y) = 2x + y$.
3. $h(x, y) = xy$.

4. $l(x, y, z) = x - y + 2z$.
5. $m(x, y, z) = x^2 + y^2$.
6. $n(x, y, z) = x^2 + y^2 + z^2$.
7. $p(x, y, z) = x^2 + y^2 - z$.

Answer: 1. $f(x, y) = x^2 + y^2$: concentric circles centered at the origin.
2. $g(x, y) = 2x + y$: lines with slope -2.
3. $h(x, y) = xy$: these should look like rotated hyperbolæ, or the graphs of $y = k/x$ for different constants k.
4. $l(x, y, z) = x - y + 2z$: planes with normal $\langle 1, -1, 2\rangle$.
5. $m(x, y, z) = x^2 + y^2$: cylinders with the z-axis as the cylindrical axis.
6. $n(x, y, z) = x^2 + y^2 + z^2$: spheres centered at the origin.
7. $p(x, y, z) = x^2 + y^2 - z$: paraboloids opening in the positive z-direction, and shifted above or below the xy plane.

8. (§2.1) Sketch the graphs of the following functions:
1. $f(x, y) = x^2 + y^2$.
2. $g(x, y) = 2x + y$.

Answer: 1. $f(x, y) = x^2 + y^2$. This is a paraboloid.
2. $g(x, y) = 2x + y$. This is a plane with normal $\langle 2, 1, -1\rangle$, and going through the origin.

9. (§2.3) Find ∇f for
1. $f(x, y) = x^2 + y^2$.
2. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$.
3. $f(x, y) = e^{x+y}$.
4. $f(x, y, z) = e^{x+y}\cos z + y^2$.

Answer: 1. $\nabla f(x, y) = \langle 2x, 2y\rangle$.
2. $\nabla f(x, y, z) = \left\langle \frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}} \right\rangle = \langle x, y, z\rangle / f(x, y, z)$.
3. $\nabla f(x, y) = \langle e^{x+y}, e^{x+y}\rangle = \langle f(x, y), f(x, y)\rangle$.
4. $\nabla f(x, y, z) = \langle e^{x+y}\cos z, e^{x+y}\cos z + 2y, -e^{x+y}\sin z\rangle$.

10. (§2.3) Find the equation of the tangent plane to the graph of f at the point P for
1. $f(x, y) = x^2 + y^2$, $P = (3, 4, 25)$.
2. $f(x, y) = e^{x+y}$, $P = \left(2, 1, e^3\right)$.

Answer: If $P = (x_0, y_0, f(x_0, y_0))$, the general form is

$$z - f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \langle x - x_0, y - y_0\rangle$$

1. The plane is $z - 25 = \langle 6, 8\rangle \cdot \langle x - 3, y - 4\rangle$. This can be simplified to the usual $ax + by + cz + d = 0$ form.
2. The plane is $z - e^3 = \left\langle e^3, e^3\right\rangle \cdot \langle x - 1, y - 1\rangle$.

11. (§2.3) In your own words, what does it mean for a function $f\colon \mathbb{R}^n \to \mathbb{R}$ to be "differentiable" at a point x_0? If f is differentiable at x_0, does f have to be smooth at this point? Is there a unique tangent plane at this point?

Answer: The function is differentiable if it is smooth around x_0. If all the parial derivatives of f exist and are continuous at x_0, then it is differentiable. Another characterization is that the difference between the function and its tangent (hyper)plane at a point x is, in the limit, "smaller" than the distance from x to x_0. Differentiability gives a certain amount of smoothness. Moreover, if f is differentiable at a point it does have a unique tangent plane. Your words may vary.

12. (§2.3) Remember the theorem: if all the partial derivatives of $f\colon \mathbb{R}^n \to \mathbb{R}$ exist and are continuous at x_0, then f is differentiable at x_0.

Answer: Noted.

13. (§2.4) What kind of curve does $\boldsymbol{c}(t)$ trace out for

1. $\boldsymbol{c}(t) = \langle 1+2t, 3, 4-t\rangle$
2. $\boldsymbol{c}(t) = \langle \cos 2t, \sin 2t\rangle$.
3. $\boldsymbol{c}(t) = \langle \cos t, \sin t, 3t\rangle$.

Answer:

1. $\boldsymbol{c}(t) = \langle 1+2t, 3, 4-t\rangle$: a straight line parallel to $\langle 2, 0, -1\rangle$.
2. $\boldsymbol{c}(t) = \langle \cos 2t, \sin 2t\rangle$: a circle in the plane centered at the origin.
3. $\boldsymbol{c}(t) = \langle \cos t, \sin t, 3t\rangle$: a helix.

14. (§2.4) Given the path $\boldsymbol{c}(t)$, find its velocity function $\boldsymbol{c}'(t)$, and its speed function $\|\boldsymbol{c}'(t)\|$

1. $\boldsymbol{c}(t) = \langle 1+3t, 3, 1-4t\rangle$
2. $\boldsymbol{c}(t) = \langle \cos t, \sin t, 3t\rangle$.
3. $\boldsymbol{c}(t) = \left\langle t^2, t^3, t\right\rangle$.
4. $\boldsymbol{c}(t) = \left\langle t, t^3 + 1/(4t)\right\rangle$.
5. $\boldsymbol{c}(t) = \left\langle t/\sqrt{2}, t/\sqrt{2}, \ln(\sec t)\right\rangle$.

Answer:

1. $\boldsymbol{c}'(t) = \langle 3, 0, -4\rangle$, and $\|\boldsymbol{c}'(t)\| = 5$.
2. $\boldsymbol{c}'(t) = \langle -\sin t, \cos t, 3\rangle$, and $\|\boldsymbol{c}'(t)\| = \sqrt{10}$.
3. $\boldsymbol{c}'(t) = \left\langle 2t, 3t^2, 1\right\rangle$, and $\|\boldsymbol{c}'(t)\| = \sqrt{4t^2 + 9t^4 + 1}$.
4. $\boldsymbol{c}'(t) = \left\langle 1, 3t^2 - 1/(4t^2)\right\rangle$, and $\|\boldsymbol{c}'(t)\| = \sqrt{1 + 9t^4 - 3/2 + 1/(16t^4)}$.
5. $\boldsymbol{c}'(t) = \left\langle 1/\sqrt{2}, 1/\sqrt{2}, \tan t\right\rangle$, and $\|\boldsymbol{c}'(t)\| = \sqrt{1+\tan^2 t} = |\sec t|$.

15. (§2.4) Given the path $\boldsymbol{c}(t)$, and a time t_0, find the tangent vector to $\boldsymbol{c}(t_0)$ for

1. $\boldsymbol{c}(t) = \langle 1+3t, 3, 1-4t\rangle\,,\ t_0 = 1$.
2. $\boldsymbol{c}(t) = \left\langle t^2, -3t, 1+t^3\right\rangle\,,\ t_0 = 2$.

Answer:

1. $\boldsymbol{c}(t) = \langle 3, 0, -4\rangle$.
2. $\boldsymbol{c}(t) = \langle 4, -3, 12\rangle$.

16. (§2.5) Given the path $\boldsymbol{c}(t)$ in $\mathbb{R}^n$ and a real valued function $f(\boldsymbol{x})$ on n variables, find $\frac{\mathrm{d}}{\mathrm{d}t}\left[f \circ \boldsymbol{c}(t)\right]$

1. $\boldsymbol{c}(t) = \langle 1+3t, 3, 1-4t\rangle\,,\ f(x,y,z) = xy - yz$.
2. $\boldsymbol{c}(t) = \langle \cos t, \sin t, t\rangle\,,\ f(x,y,z) = x^2 + y^2 - z$
3. $\boldsymbol{c}(t) = \left\langle t, t^2, t^3\right\rangle\,,\ f(x,y,z) = xyz + e^{xy}$.

Answer: You *could* do this by substitution and use the chain rule from M20A. However, in this class we should use the multivariable chain rule:

$$\frac{\mathrm{d}}{\mathrm{d}t}[f \circ \boldsymbol{c}(t)] = \nabla f(\boldsymbol{c}(t)) \cdot \boldsymbol{c}'(t)$$

The difficult thing is to remember to plug $\vec{c}(t)$ into f as the x, y and z.

1. We get $\nabla f(x,y,z) = \langle y, x-z, -y\rangle$, and $\boldsymbol{c}'(t) = \langle 3, 0, -4\rangle$. So

$$\frac{\mathrm{d}}{\mathrm{d}t}[f \circ \boldsymbol{c}(t)] = \langle 3, 1+3t-(1-4t), -3\rangle \cdot \langle 3, 0, -4\rangle = 9 + 12 = 21$$

2. We get $\nabla f(x,y,z) = \langle 2x, 2y, -1\rangle$, and $\boldsymbol{c}'(t) = \langle -\sin t, \cos t, 1\rangle$. So

$$\frac{\mathrm{d}}{\mathrm{d}t}[f \circ \boldsymbol{c}(t)] = \langle -2\sin t, 2\cos t, 1\rangle \cdot \langle -\sin t, \cos t, 1\rangle = 2\sin^2 t + 2\cos^2 t + 1 = 3$$

17. (§2.6) Given the real valued function $f(\boldsymbol{x})\colon \mathbb{R}^n \to \mathbb{R}$, and the vector $\boldsymbol{v}$, find the directional derivative of f along $\boldsymbol{v}$ at a given point, for

1. $f(x,y) = x^2 + y^2$, $\boldsymbol{v} = \langle 1, 3\rangle$, at the point $(3,4)$.
2. $f(x,y,z) = \sqrt{x^2+y^2+z^2}$, $\boldsymbol{v} = \langle -1, 0.5, 3\rangle$, at the point $(1,0,-1)$.
3. $f(x,y) = e^{x+y}$, $\boldsymbol{v} = \left\langle \sqrt{2}/2, \sqrt{2}/2\right\rangle$, at the point $(1,-1)$.
4. $f(x,y,z) = e^{x+y}\cos z + y^2$, $\boldsymbol{v} = \left\langle 1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}\right\rangle$, at the point $(2,1,\pi/2)$.
5. $f(x,y,z) = x + y\ln z$, $\boldsymbol{v} = \langle 1/3, 1, -1\rangle$ at the point $(1,2,0)$.

Answer: By directional derivative "along" $\boldsymbol{v}$, I mean $\nabla f \cdot \boldsymbol{v}$. When we talk about the directional derivative "in the direction of" $\boldsymbol{v}$, we mean $\nabla f \cdot (\boldsymbol{v}/\|\boldsymbol{v}\|)$. So these are easier to do these when you've already computed the gradient, as we have for many of these functions in problem 9.

1. $\nabla f(x,y) = \langle 2x, 2y\rangle$, and $\nabla f(3,4) = \langle 6, 8\rangle$, so the directional derivative is $\langle 6,8\rangle \cdot \langle 1,3\rangle = 6 + 24 = 30$. The meaning of this number is that a particle at $(3,4)$ moving with velocity $\boldsymbol{v}$ experiences an instantaneous change of f of 30 units/second.
2. $\nabla f(x,y,z) = \langle 2x, 2y, 2z\rangle$, and $\nabla f(1,0,-1) = \langle 2, 0, -2\rangle$, so the directional derivative is $\langle 2,0,-2\rangle \cdot \langle -1, 0.5, 3\rangle = -2 + 6 = 4$.
3. $\nabla f(x,y) = \langle e^{x+y}, e^{x+y}\rangle$, and $\nabla f(1,-1) = \langle 1, 1\rangle$, so the directional derivative along $\boldsymbol{v}$ is $\langle 1,1\rangle \cdot \left\langle \sqrt{2}/2, \sqrt{2}/2\right\rangle = \sqrt{2}$.
4. $\nabla f(x,y,z) = \langle e^{x+y}\cos z, e^{x+y}\cos z + 2y, -e^{x+y}\sin z\rangle$, and $\nabla f(2,1,\pi/2) = \left\langle 0, 2, -e^3\right\rangle$, so the directional derivative along $\boldsymbol{v}$ is $\left\langle 0,2,-e^3\right\rangle \cdot \left\langle 1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}\right\rangle = \left(2 - e^3\right)/\sqrt{3}$

18. (§2.6) Given the real valued function $f(\boldsymbol{x})\colon \mathbb{R}^n \to \mathbb{R}$, and a point in $\mathbb{R}^n$, find the direction (*i.e.*, unit vector) of maximal increase of f at that point for

1. $f(x,y) = x^2 + y^2$, at the point $(3,4)$.
2. $f(x,y) = xy + 1/(xy)$, at the point $(1,2)$.
3. $f(x,y,z) = xyz$, at the point $(1,0,2)$.
4. $f(x,y,z) = 2x + 3y - 4z$, at the point $(1,-1,5)$.
5. $f(x,y,z) = x^2 + y^2 - z$, at the point $(0,1,1)$.

Answer: The "direction of maximal increase of f" is a vector in the same direction as ∇f. So this becomes a problem of computing the gradient and then making a unit length vector in the same direction.

1. $\nabla f(x, y) = \langle 2x, 2y \rangle$, and $\nabla f(3, 4) = \langle 6, 8 \rangle$, so the direction of maximal increase is $\langle 6, 8 \rangle / \sqrt{36 + 64} = \langle 6/10, 8/10 \rangle$.
2. $\nabla f(x, y) = \left\langle y - 1/(x^2 y), x - 1/(xy^2) \right\rangle$, and $\nabla f(1, 2) = \langle 3/2, 3/4 \rangle$ The direction of maximal increase, then, is $\langle 3/2, 3/4 \rangle / \sqrt{9/4 + 9/16} = (4/\sqrt{45}) \langle 3/2, 3/4 \rangle = \left(1/\sqrt{45}\right) \langle 6, 3 \rangle$.
3. $\nabla f(x, y, z) = \langle yz, xz, xy \rangle$, and $\nabla f(1, 0, 2) = \langle 0, 2, 0 \rangle$ so the direction of maximal increase is $\langle 0, 1, 0 \rangle$.

19. (§3.2) Find the linear Taylor formula (*i.e.*, linearization) of $f(x, y) = x^2 + y^2$ at the point $(1, 2)$.

Answer: We have

$$\begin{aligned} f\left((1,2) + (h_1, h_2)\right) &\approx f(1,2) + \nabla f(1,2) \cdot \langle h_1, h_2 \rangle \\ &= \left(1^2 + 2^2\right) + \langle 2, 4 \rangle \cdot \langle h_1, h_2 \rangle \\ &= 5 + 2h_1 + 4h_2 \end{aligned}$$

You can turn this into an equation involving (x, y) by letting $x = 1 + h_1$, and $y = 2 + h_2$. This gives

$$f(x, y) \approx 2x + 4y - 10,$$

the graph of which is a plane–the tangent plane of $f(x, y)$ at $(1, 2)$.

20. (§3.2) Find the quadratic Taylor formula of $f(x, y) = x^2 + y^2$ at the point $(1, 2)$.

Answer: We add a quadratic term on to the linear formula:

$$\begin{aligned} f\left((1,2) + (h_1, h_2)\right) &\approx f(1,2) + \nabla f(1,2) \cdot \langle h_1, h_2 \rangle \\ &\quad + \frac{1}{2}\left[\frac{\partial^2 f}{\partial x^2}(1,2)\, h_1^2 + \frac{\partial^2 f}{\partial x \partial y}(1,2)\, h_1 h_2 + \frac{\partial^2 f}{\partial y \partial x}(1,2)\, h_2 h_1 + \frac{\partial^2 f}{\partial y^2}(1,2)\, h_2^2\right] \\ &= 5 + 2h_1 + 4h_2 + \frac{1}{2}\left[2h_1^2 + 0h_1 h_2 + 0h_2 h_1 + 2h_2^2\right] \\ &= 5 + 2h_1 + 4h_2 + h_1^2 + h_2^2. \end{aligned}$$

21. (§4.2) Given path $\boldsymbol{c}(t)$, find the arc length of the path for t between t_0 and t_1, for

1. $\boldsymbol{c}(t) = \langle 1 + 3t, 3, 1 - 4t \rangle$, $t_0 = 0$, $t_1 = 3$.
2. $\boldsymbol{c}(t) = \langle \cos t, \sin t, 3t \rangle$, $t_0 = 0$, $t_1 = 4\pi$.
3. $\boldsymbol{c}(t) = \left\langle t/\sqrt{2}, t/\sqrt{2}, \ln(\sec t) \right\rangle$, $t_0 = 0$, $t_1 = \pi/3$.
4. $\boldsymbol{c}(t) = \left\langle t, t^3/3 + 1/(4t) \right\rangle$, $t_0 = 1$, $t_1 = 2$.

Answer: Solveable problems of this type are few and far between. These examples pretty much exhaust my supply of good problems. We examined most of these paths in problem 14. Thus we don't have to recompute the speed. Recall that the arc length of $\boldsymbol{c}(t)$ for $t_0 \leq t \leq t_1$ is

$$\int_{t_0}^{t_1} \left\| \boldsymbol{c}'(t) \right\| \, \mathrm{d}t$$

1. $\int_0^3 5\,\mathrm{d}t = 15$
2. $\int_0^{4\pi} \sqrt{10}\,\mathrm{d}t = 4\pi\sqrt{10}$.
3. $\int_0^{\pi/3} \sec t\,\mathrm{d}t = \ln(\sec t + \cos t)\Big|_0^{\pi/3} = 2 + \sqrt{3} - 1 = 1 + \sqrt{3}$.
4. This one didn't appear correctly in problem 14, so we compute $\boldsymbol{c}'(t) = \left\langle 1, t^2 - 1/(4t^2)\right\rangle$. Now note that

$$\begin{aligned}\|\boldsymbol{c}'(t)\| &= \sqrt{1 + (t^2 - 1/(4t^2))^2} = \sqrt{1 + t^4 - 1/2 + 1/(16t^4)} \\ &= \sqrt{t^4 + 1/2 + 1/(16t^4)} = \sqrt{(t^2 + 1/(4t^2))^2} = t^2 + 1/(4t^2)\end{aligned}$$

Thus the answer is

$$\int_1^2 t^2 + 1/(4t^2)\,\mathrm{d}t = t^3/3 - 1/(4t)\Big|_1^2 = 8/3 - 1/8 - (1/3 - 1/4) = 7/3 + 1/8$$

22. (§4.4) Match four of the following vector fields to their graphical representation in Figure 1. (Two of the following fields are not plotted.)

1. $\boldsymbol{F}(x, y) = \langle y, x\rangle$.
2. $\boldsymbol{F}(x, y) = \left\langle x^2, y^2\right\rangle$.
3. $\boldsymbol{F}(x, y) = \langle 2, 5\rangle$.
4. $\boldsymbol{F}(x, y) = \langle y, -x\rangle$.
5. $\boldsymbol{F}(x, y) = \langle 2x, 2y\rangle$.
6. $\boldsymbol{F}(x, y) = \langle -y, x\rangle$.

Answer: One way to do this is to sketch each of these vector fields. By "sketch" I mean pick a few points and compute the vector field at those points and see what you get. You will find that (a) is $\langle 2x, 2y\rangle$, (b) is $\langle 2, 5\rangle$, (c) is $\langle y, x\rangle$, and (d) is $\langle -y, x\rangle$.

23. (§4.4) Draw some flow lines for the vector fields in Figure 1.

Answer: You do that.

24. (§4.4) Which of the vector fields plotted in Figure 1 might be gradient fields, that is of the form $\nabla\phi$ for some scalar field ϕ? For those that are, try to draw some level sets for the function ϕ over the vector field.

Answer: In (a) the field is the gradient of $\phi(x, y) = x^2 + y^2$. The level sets are concentric circles centered at the origin. In (b) the field is the gradient of $\phi(x, y) = 2x + 5y$. The level sets are straight lines with slope $-2/5$. In (c) the field is the gradient of $\phi(x, y) = xy$. The level sets are those hyperbola like things (see my lame description of these in problem 7). The field in (d) is not a gradient. If you try to draw a level set you will find they are lines going through the origin, and thus they all have the same value and thus ϕ has to be a constant and its gradient would have to be the field of all zeros, a contradiction.

25. (§4.4) Given the vector field $\boldsymbol{F}(x, y, z)$, find $\operatorname{div}\boldsymbol{F} = \nabla\cdot\boldsymbol{F}$, and $\mathbf{curl}\,\boldsymbol{F} = \nabla\times\boldsymbol{F}$ for

1. $\boldsymbol{F}(x, y, z) = \left\langle xy + z, x^2 - y, z^2 - 3\right\rangle$.
2. $\boldsymbol{F}(x, y, z) = \langle yz, xz, xy\rangle$.
3. $\boldsymbol{F}(x, y, z) = \langle 2x, 2y, -1\rangle$.

4. $\boldsymbol{F}(x,y,z) = \langle y, -x, 0\rangle$.
5. $\boldsymbol{F} = \nabla\phi$, where $\phi(x,y,z) = 2xy^2 - ze^y$.
6. $\boldsymbol{F} = \nabla \times \boldsymbol{G}$, where $\boldsymbol{G}(x,y,z) = \left\langle xy+z, x^2-y, z^2-3\right\rangle$.

Answer: 1. $\boldsymbol{F} = \left\langle xy+z, x^2-y, z^2-3\right\rangle$: $\nabla \cdot \boldsymbol{F}(x,y,z) = y + -1 + 2z$. $\nabla \times \boldsymbol{F}(x,y,z) = \langle 0, 1, x\rangle$

2. $\boldsymbol{F} = \langle yz, xz, xy\rangle$: $\nabla \cdot \boldsymbol{F} = 0$. $\nabla \times \boldsymbol{F} = \mathbf{0}$.
3. $\boldsymbol{F} = \langle 2x, 2y, -1\rangle$: $\nabla \cdot \boldsymbol{F} = 4$. $\nabla \times \boldsymbol{F} = \mathbf{0}$.
4. $\boldsymbol{F} = \langle y, -x, 0\rangle$: $\nabla \cdot \boldsymbol{F} = 0$. $\nabla \times \boldsymbol{F} = \langle 0, 0, -2\rangle$.
5. $\boldsymbol{F} = \nabla\phi$, where $\phi(x,y,z) = 2xy^2 - ze^y$: in this case $\nabla\cdot\boldsymbol{F} = \nabla\cdot\nabla\phi$. This is sometimes written as $\nabla^2\phi$, or, more confusingly, as $\Delta\phi$. This is known as the *Laplacian* of ϕ. It turns out to be the sum of the second partials. That is $\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$. In this case we get $\nabla^2\phi = 0 + (4x - ze^y) + 0$. There was nothing stopping you, however, from first evaluating $\boldsymbol{F}$, then taking its divergence–you should get the same answer. The curl of this $\boldsymbol{F}$ is $\mathbf{0}$ because we have the rule that $\nabla \times \nabla\phi = \mathbf{0}$ for ϕ a function with continuous second derivatives.
6. $\boldsymbol{F} = \nabla \times \boldsymbol{G}$, where $\boldsymbol{G}(x,y,z) = \left\langle xy+z, x^2-y, z^2-3\right\rangle$.
   We have a rule that $\nabla \cdot (\nabla \times \boldsymbol{G}) = 0$ for a field whose components have continuous second partial derivatives, like this one. So the divergence is zero. Alternatively we could evaluate $\boldsymbol{F}$ and then take its divergence. We did this above in an earlier part of this problem, and have $\boldsymbol{F} = \langle 0, 1, x\rangle$. Taking the divergence indeed gives 0. Now to the curl we evaluate $\nabla \times \langle 0, 1, x\rangle = \langle 0, 1, 0\rangle$.

Exam 2 Prep Sheet:

26. (§5.1–5.4) Given function $f(x,y)$ and region $D \subseteq \mathbb{R}^2$, evaluate

$$\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

1. $f(x,y) = 2xy$, D is the region bounded by $y = 0, x = 2, y = x^2$.
2. $f(x,y) = y^2\cos x$, D is region bounded by $x = y^3, y = -1, y = 1, x = 3$.

Answer: These problems are really review from 20C.

1. I get

$$\int_0^2 \int_0^{x^2} 2xy\,\mathrm{d}y\,\mathrm{d}x = 32/3$$

2. This is

$$\int_{-1}^1 \int_{y^3}^3 y^2\cos x\,\mathrm{d}x\,\mathrm{d}y = \cos(1)$$

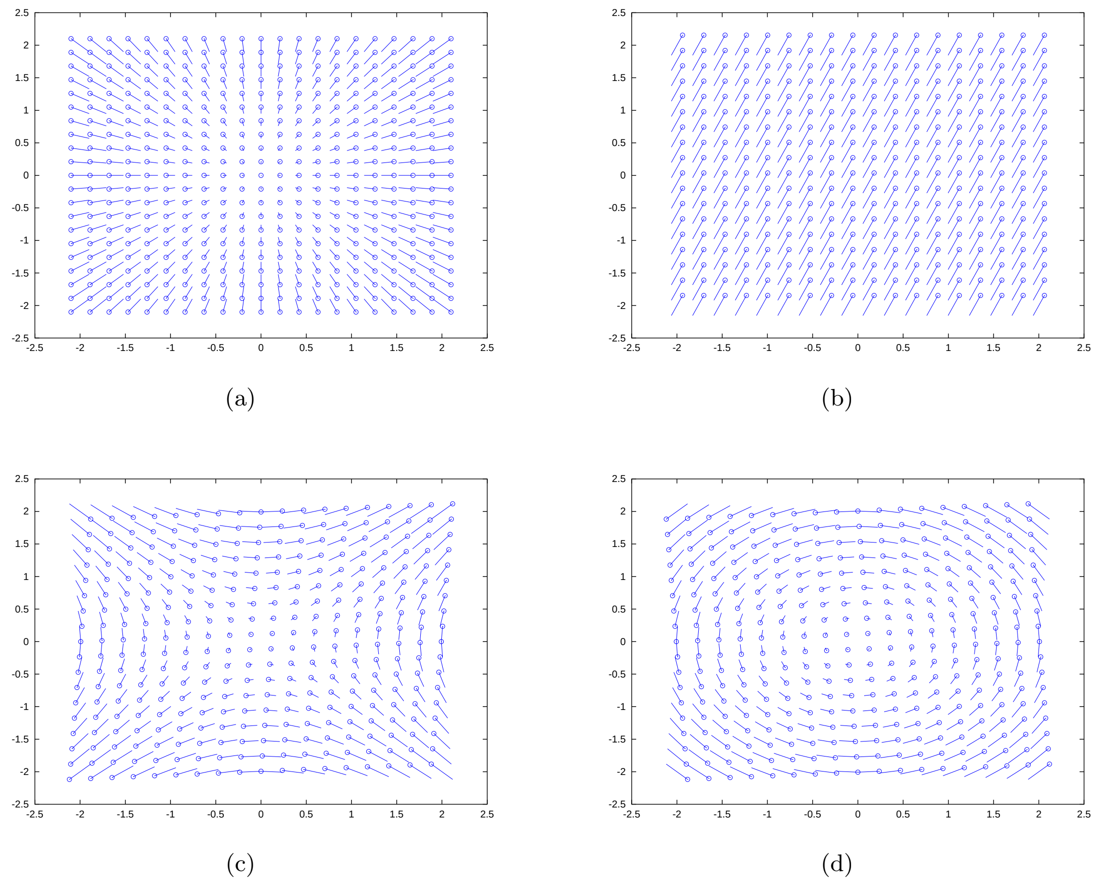

Figure 1: Four vector fields. Circles represent the tips of the vectors (proper arrowheads were not available).

27. (§5.1–5.4) Given function $f(x, y, z)$ and region $D \subseteq \mathbb{R}^3$, evaluate

$$\iiint_D f(x, y, z)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$$

1. $f(x, y, z) = \sqrt{x^2 + z^2}$, D is bounded by $y = x^2 + z^2, y = 4$.
2. $f(x, y, z) = x + y + z$, D is bounded by $x = y^2, x = z, z = 0, x = 1$.

Answer: 1. Here the region D is a paraboloid, but because the coordinates have been switched, it seems more dificult. I think the answer is $\frac{256\pi}{45}\sqrt{2}$.

2. It is a bit tricky to visualize D. My answer was $6/7$.

28. (§6.1) Given a change of variables transformation $T\colon D^* \subseteq \mathbb{R}^n \to \mathbb{R}^n$, and the domain, D^*, find the image of the transformation, $D = T(D^*)$. Is the transformation one-to-one?

1. $(x, y) = T(r, \theta) = (r\cos\theta, r\sin\theta)$, and $D^* = [0, 4] \times [0, \pi/2]$.

2. $(x, y, z) = T(r, \theta, z) = (r\cos\theta, r\sin\theta, z)$, and $D^* = [0,1] \times [0,\pi] \times [0,1]$.

Answer: 1. D is the part of a disc of radius 4 in the first quadrant. The mapping is not one-to-one on D^* because $T(0,\theta) = (0,0)$ for any θ. However, the mapping *is* one-to-one on the interior of D^*. By "interior," I mean $(0,4) \times (0,\pi/2)$.

2. D is half of a solid cylinder of radius 1 and height 1. It is the part of this cylinder in the halfspace $y \geq 0$. Again, the mapping is not one-to-one on D^* because it takes many points to the z-axis.

29. (§6.1) Given a change of variables transformation $T\colon D^* \subseteq \mathbb{R}^n \to \mathbb{R}^n$, and a set D, find a domain, D^*, such that $D = T(D^*)$. Is the transformation one-to-one?

1. $(x, y) = T(u, v) = (u/v, v)$, and D is the region bounded by the curves $xy = 1, xy = 4, y = 1, y = 3$.

Answer: 1. A good way to find the "preimage" of a set under a transformation is to "pull-back" the equation of its boundaries. In this case, the region is bounded by the curves $xy = 1, xy = 4, y = 1, y = 3$. If we write these in terms of u and v, we will find curves which bound D^*. Thus under the transfrom $(x, y) = (u/v, v)$, the equation $xy = 1$ becomes $(u/v)v = 1$, or $u = 1$. The three other curves are handled similarly to give $D^* = [1,4] \times [1,3]$. This transformation is one-to-one.

30. (§6.2) Given a change of variables transformation $T\colon D^* \subseteq \mathbb{R}^n \to \mathbb{R}^n$, find the Jacobian Determinant of the transformation.

1. With $(x, y) = T(r, \theta) = (r\cos\theta, r\sin\theta)$, find $\frac{\partial(x,y)}{\partial(r,\theta)}$.
2. With $(x, y) = T(u, v) = (u/v, v)$, find $\frac{\partial(x,y)}{\partial(u,v)}$.
3. With $(x, y) = T(u, v) = ((u+v)/2, (u-v)/2)$, find $\frac{\partial(x,y)}{\partial(u,v)}$.
4. With $(x, y) = T(u, v) = (au + bv + c, du + fv + g)$, find $\frac{\partial(x,y)}{\partial(u,v)}$.
5. With $(x, y, z) = T(u, v, w) = (au, bv, cw)$, find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$.
6. With $(x, y, z) = T(r, \theta, z) = (r\cos\theta, r\sin\theta, z)$, find $\frac{\partial(x,y,z)}{\partial(r,\theta,z)}$.
7. With $(x, y, z) = T(\rho, \theta, \phi) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$, find $\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}$

Answer: Remember that

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

1. $\frac{\partial(x,y)}{\partial(r,\theta)} = r$.
2. $\frac{\partial(x,y)}{\partial(u,v)} = 1/v$.
3. $\frac{\partial(x,y)}{\partial(u,v)} = -1$.
4. $\frac{\partial(x,y)}{\partial(u,v)} = af - bd$.
5. $\frac{\partial(x,y,z)}{\partial(u,v,w)} = abc$.
6. $\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = r$.
7. $\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \rho^2\sin\phi$

31. (§6.2) Given a change of variables transformation $T\colon D^* \subseteq \mathbb{R}^n \to \mathbb{R}^n$, some set D, and a function $f(x,y)$, evaluate

$$\iint\limits_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y \quad \text{as an integral of the form} \quad \iint\limits_{D^*} f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\,\mathrm{d}v$$

You may be forced to find D^* yourself.

1. $f(x,y) = \sin\left(x^2+2xy+y^2\right)$, D is the triangle with corners $(0,0)$,$(1,0)$,$(0,1)$ using the change of variables $x = (u+v)/2, y = (u-v)/2$.
2. $f(x,y) = e^{xy}$, D is bounded by $xy = 1, xy = 4, y = 1, y = 3$, and using the change of variables $x = u/v, y = v$.

Answer: 1. The set D is bounded by $x = 0$, $y = 0$, and $y = 1 - x$. Using the given transformation, these become $u+v = 0$, $u-v = 0$, and $u = 1$. The integral becomes

$$\iint\limits_{D^*} \sin\left(u^2\right)\left|-\frac{1}{2}\right| \mathrm{d}A = \frac{1}{2}\int_0^1\int_{-u}^{u} \sin\left(u^2\right)\,\mathrm{d}v\,\mathrm{d}u = -\frac{1}{2}\cos(1)+\frac{1}{2}$$

2. See Problem 29 to find D^*.
$\left(e^4 - e\right)\ln 3$.

32. (§7.1) Given the scalar field $f(x,y,z)$ and path $\boldsymbol{c}(t)\colon [a,b] \to \mathbb{R}^3$ evaluate the path integral

$$\int\limits_{\boldsymbol{c}} f\,\mathrm{d}s$$

1. $f(x,y,z) = x^2+y^2$, along $\boldsymbol{c}(t) = \langle r\cos t, r\sin t, t\rangle$ for $a \le t \le b$.
2. $f(x,y,z) = z$, along $\boldsymbol{c}(t) = \langle \cos t, \sin t, t\rangle$ for $0 \le t \le 2\pi$.
3. $f(x,y,z) = \cos\left(\sqrt{z}+\sqrt{y}\right)$, along $\boldsymbol{c}(t) = \left\langle 1, t^2/4, t^2/4\right\rangle$ for $0 \le t \le 1$.

Answer: There is no shortcut for path integrals.

1. First, $\|\boldsymbol{c}'(t)\| = \sqrt{r^2+1}$. So you get

$$\int_a^b r^2\sqrt{r^2+1}\,\mathrm{d}t = (b-a)\,r^2\sqrt{1+r^2}$$

2. I get $\int_0^{2\pi} t\sqrt{2}\,\mathrm{d}t = 2\sqrt{2}\pi^2$.
3. $\int_0^1 \cos t\left(t/\sqrt{2}\right)\,\mathrm{d}t$, which you solve by parts to get $(\sin 1 + \cos 1 - 1)/\sqrt{2}$.

33. (§7.2) Given the vector field $\boldsymbol{F}(x,y,z)$ and path $\boldsymbol{c}(t)\colon [a,b] \to \mathbb{R}^3$ evaluate the line integral

$$\int\limits_{\boldsymbol{c}} \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{s}$$

1. $\boldsymbol{F} = \left\langle 3x+4y, 2x+3y^2\right\rangle$, along $\boldsymbol{c}(t) = \langle 2\cos t, 2\sin t\rangle$, $0 \le t \le 2\pi$.

2. $\boldsymbol{F} = \left\langle 2xy, x^2+z, y \right\rangle$, along the path which traces a straight line from $(1,0,2)$ to $(3,4,1)$.
3. $\boldsymbol{F} = \langle yz, xz, xy \rangle$ along some path $\boldsymbol{c}$ with $\boldsymbol{c}(a) = (-1,-1,2)$, and $\boldsymbol{c}(b) = (-2,1,-1)$.
4. $\boldsymbol{F} = \left\langle 2xz + y^2z^2, 2xyz^2, x^2 + 2xy^2z \right\rangle$ along the path $\boldsymbol{c}(t) = \left\langle \cos(\pi t) + e^t, -\sin(\pi t) + \log(1+t), t^{12} \right\rangle$ for $0 \le t \le 1$.

Answer: Some of these use the following fact (which is one of the reasons we care about conservative vector fields): if $\boldsymbol{F}$ is conservative, *i.e.*, $\boldsymbol{F} = \nabla\phi$ for a scalar field ϕ, then

$$\int_{\boldsymbol{c}} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = \int_a^b \nabla\phi(\boldsymbol{c}(t)) \cdot \boldsymbol{c}'(t)\,\mathrm{d}t = \phi(\boldsymbol{c}(t))\Big|_a^b = \phi(\boldsymbol{c}(b)) - \phi(\boldsymbol{c}(a))$$

We should use this shortcut when $\boldsymbol{c}$ is given ambiguously, or when the problem would be too horendous to do the long way.

1. We must do this directly, since this field is not conservative. The answer requires tedious calculations involving trigonometry:

$$\int_0^{2\pi} \left\langle 6\cos t + 8\sin t, 4\cos t + 12\sin^2 t \right\rangle \cdot \langle -2\sin t, 2\cos t \rangle\ \mathrm{d}t$$
$$= \int_0^{2\pi} -4\sin t\cos t - 16\sin^2 t + 24\sin^2 t\cos t\,\mathrm{d}t\ldots$$

2. 40. Can be done directly or by using a potential.
3. 0. The path is given only vaguely, so we must use the fact that the field is conservative: $\boldsymbol{F} = \nabla\phi$ with $\phi = xyz$.
4. This problem is so horrendous, you probably cannot do it except by using the potential of the field, $\phi = x^2z + xy^2z^2$. The answer is $(e-1)\left[e - 1 + (\log 2)^2\right]$.

34. (§7.2) Given a vector field $\boldsymbol{F}$, determine if it is conservative. If it is conservative, find a ϕ such that $\boldsymbol{F} = \nabla\phi$.

1. $\boldsymbol{F} = \left\langle 2xy, x^2 + z, y \right\rangle$.
2. $\boldsymbol{F} = \langle xy, z, x \rangle$.
3. $\boldsymbol{F} = \left\langle x^2y + 1, \frac{1}{3}x^3 + 1, y \right\rangle$.
4. $\boldsymbol{F} = \frac{1}{x^2+y^2+z^2} \langle x, y, z \rangle$.
5. $\boldsymbol{F} = \langle -\sin x\cos y, \cos x\cos y, 1 \rangle$.
6. $\boldsymbol{F} = \frac{1}{(x^2+y^2+z^2)^{3/2}} \langle x, y, z \rangle$.

Answer: For these assume that $\boldsymbol{F}$ *is* conservative and try to find a potential, ϕ. Do this by integrating with respect to each of the variables:

1. Assuming $\boldsymbol{F} = \nabla\phi$, we have

$$\frac{\partial \phi}{\partial x} = 2xy \Rightarrow \phi = x^2y + C_1(y,z)$$
$$\frac{\partial \phi}{\partial y} = x^2 + z \Rightarrow \phi = x^2y + zy + C_2(x,z)$$
$$\frac{\partial \phi}{\partial z} = y \Rightarrow \phi = zy + C_3(x,y)$$

These are satisfied by $C_1(y,z) = zy + C$, $C_2(x,z) = C$, $C_3(x,y) = x^2y + C$, for any C. Thus we can choose $\phi = x^2y + zy$, and the field is conservative.

2. Assuming $\boldsymbol{F} = \nabla\phi$, we have

$$\frac{\partial \phi}{\partial x} = xy \Rightarrow \phi = x^2y + C_1(y,z)$$
$$\frac{\partial \phi}{\partial y} = z \Rightarrow \phi = zy + C_2(x,z)$$
$$\frac{\partial \phi}{\partial z} = x \Rightarrow \phi = zx + C_3(x,y)$$

There is no consistent choice of C_1, C_2, C_3 that gives a single scalar field ϕ, since, for example, $C_2(x,z)$ would have to contain the term x^2y, and it cannot since it is a function of only x and z. Thus this field is not conservative.

3. not conservative.
4. $\phi = \frac{1}{2}\ln\left(x^2+y^2+z^2\right) + C$.
5. not conservative.
6. $\phi = -\frac{1}{\sqrt{x^2+y^2+z^2}} + C$.

35. (§7.3) Given a parametrization of a surface, $\mathbf{\Phi}(u,v)$, find $\boldsymbol{T}_u \times \boldsymbol{T}_v$. Is the parametrization regular?

1. $\mathbf{\Phi}(u,v) = \left\langle v\cos u, v\sin u, v^2\right\rangle$.
2. $\mathbf{\Phi}(u,v) = \left\langle (1+\cos u)\cos v, (1+\cos u)\sin v, \sin v\right\rangle$.
3. $\mathbf{\Phi}(u,v) = \left\langle u, v, g(u,v)\right\rangle$.
4. $\mathbf{\Phi}(u,v) = \left\langle u, v, u^2 + v^{3/2}\right\rangle$.
5. $\mathbf{\Phi}(u,v) = \left\langle u^2, v-u, vu\right\rangle$.

Answer: 1. $\boldsymbol{T}_u \times \boldsymbol{T}_v = \left\langle -2v^2\cos u, -2v^2\sin u, v\right\rangle$. This surface is regular where $v \neq 0$.

2. $\boldsymbol{T}_u \times \boldsymbol{T}_v = \left\langle -\sin u \sin v \cos v, \sin u \cos^2 v, (1+\cos u)\sin u\right\rangle$. This surface is regular where $\sin u \neq 0$.
3. $\boldsymbol{T}_u \times \boldsymbol{T}_v = \left\langle -\frac{\partial g}{\partial u}, -\frac{\partial g}{\partial v}, 1\right\rangle$. This is regular where ∇g is defined.
4. $\boldsymbol{T}_u \times \boldsymbol{T}_v = \langle -2u, -3/2\sqrt{v}, 1\rangle$. This is regular for $v \geq 0$. If $v < 0$, then $\mathbf{\Phi}(u,v)$ is not defined, and neither is $\boldsymbol{T}_u \times \boldsymbol{T}_v$.
5. $\boldsymbol{T}_u \times \boldsymbol{T}_v = \left\langle -u-v, -2u^2, 2u\right\rangle$. This is regular except for $(u,v) = (0,0)$.

36. (§7.4) Given some surface, S, find a regular parametrization $\mathbf{\Phi}(u,v)$, and domain $D \subseteq \mathbb{R}^2$ such that $S = \mathbf{\Phi}(D)$. Find $\mathrm{d}\boldsymbol{S}$ and $\mathrm{d}S = \|\mathrm{d}\boldsymbol{S}\|$. Set up an integral for the surface area of S.

1. S is the part of the cone $x^2 = y^2 + z^2$ inside the sphere $(x-8)^2 + y^2 + z^2 = 49$.
2. S is the part of the paraboloid $z = x^2 + y^2$ above the xy plane and below the plane $x + y + z = 9$.

Answer: 1. First look at the geometry of the situation: if (x, y, z) is "inside" the sphere $(x-8)^2 + y^2 + z^2 = 49$, then we should have $(x-8)^2 + y^2 + z^2 \le 49$. Moreover, if (x, y, z) is on the surface of the cone, then $x^2 = y^2 + z^2$. Merging this into the inequality gives $(x-8)^2 + x^2 \le 49$, *i.e.*, $2x^2 - 16x + 15 \le 0$. This is a classic precalculus problem, with solution

$$4 - \sqrt{17/2} \le x \le 4 + \sqrt{17/2}$$

Now parametrize in terms of r and θ (we could call them u and v, but r and θ should be more familiar). This gives $\mathbf{\Phi}(r, \theta) = \langle r, r\cos\theta, r\sin\theta \rangle$, for the region

$$D = \left\{ (r, \theta) \,\middle|\, 4 - \sqrt{17/2} \le r \le 4 + \sqrt{17/2}, 0 \le \theta \le 2\pi \right\}.$$

We have $\mathrm{d}\boldsymbol{S} = \langle r, -r\cos\theta, -r\sin\theta \rangle \, \mathrm{d}r \, \mathrm{d}\theta$, and $\mathrm{d}S = \sqrt{2} r \, \mathrm{d}r \, \mathrm{d}\theta$. The surface area is

$$A(S) = \iint_D \mathrm{d}S = \int_0^{2\pi} \int_{4-\sqrt{17/2}}^{4+\sqrt{17/2}} \sqrt{2} r \, \mathrm{d}r \, \mathrm{d}\theta = 16\pi\sqrt{17}$$

2. We did something like this in class. We have $\mathbf{\Phi}(u, v) = \left\langle u, v, u^2 + v^2 \right\rangle$, with

$$D = \left\{ (u, v) \,\middle|\, (u + 1/2)^2 + (v + 1/2)^2 = 19/2 \right\}$$

This is the graph of a function so we have $\mathrm{d}\boldsymbol{S} = \langle -2u, -2v, 1 \rangle \, \mathrm{d}u \, \mathrm{d}v$, and $\mathrm{d}S = \sqrt{1 + 4u^2 + 4v^2} \, \mathrm{d}u \, \mathrm{d}v$. We calculate the surface area as

$$A(S) = \iint_D \mathrm{d}S = \iint_D \sqrt{1 + 4u^2 + 4v^2} \, \mathrm{d}u \, \mathrm{d}v.$$

This is a hard integral.

37. (§7.5) Given some oriented surface, S, parametrized by $\mathbf{\Phi}$, and some scalar field f, find the integral of f over S:

$$\iint_S f \, \mathrm{d}S$$

1. $f(x, y, z) = 3x^2$ over the sphere of radius r.
2. $f(x, y, z) = 240xy$ over S, which is the paraboloid $z = x^2 + y^2$ for $x \in [0, 1]$, $y \in [0, 1]$.
3. $f(x, y, z) = yz$ over S, which is the boundary of the cube $[0, 1] \times [0, 1] \times [0, 1]$. That is, the cube with corners $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 1, 0)$, $(0, 1, 1)$, $(1, 0, 1)$, $(1, 1, 1)$.
4. $f(x, y, z) = 5$ over S, which is the plane $2x + 3y + 6z - 2 = 0$, with $x \ge 0$, $y \ge 0$, $z \ge 0$.

Answer: 1. Using the parametrization $\mathbf{\Phi}(\phi,\theta) = \langle r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi\rangle$, we get

$$\boldsymbol{T}_\phi \times \boldsymbol{T}_\theta = r\sin\phi\,\langle r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi\rangle$$

So $\mathrm{d}S = r^2\sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta$. This gives

$$\int_0^{2\pi}\int_0^{\pi} 3\,(r\sin\phi\cos\theta)^2\, r^2\sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta = 4\pi r^4.$$

2. Because of the limits in x and y, you should use a graph parametrization. You get

$$\int_0^1\int_0^1 240xy\sqrt{1+4x^2+4y^2}\,\mathrm{d}x\,\mathrm{d}y = 244 - 2\,(5)^{5/2}$$

3. The answer is something like

$$2\int_0^1\int_0^1 yz\,\mathrm{d}y\,\mathrm{d}z + \int_0^1\int_0^1 y\,\mathrm{d}y\,\mathrm{d}x + \int_0^1\int_0^1 z\,\mathrm{d}z\,\mathrm{d}x = \frac{1}{2}+\frac{1}{2}+\frac{1}{2} = \frac{3}{2}$$

4. The plane has normal $\boldsymbol{n} = \langle 2,3,6\rangle$. Then $\|\boldsymbol{n}\|\cos\theta = \boldsymbol{n}\cdot\boldsymbol{k} = 6$. Thus $(\cos\theta)^{-1} = \sqrt{4+9+36}/6 = 7/6$. To find the region D in the $x-y$ plane, set $z=0$ in the planar equation to get the line $2x+3y-2=0$. Thus we get

$$\int_0^1\int_0^{2(1-x)/3} 5\,\frac{7}{6}\,\mathrm{d}y\,\mathrm{d}x = \frac{35}{18}$$

38. (§7.6) Given some oriented surface, S, parametrized by $\mathbf{\Phi}$, and some vector field $\boldsymbol{F}$, find the surface integral of $\boldsymbol{F}$,

$$\iint_S \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{S}$$

1. $\boldsymbol{F} = \langle x^2, y, z^3\rangle$, S is the surface of the cube bounded by $x=0, x=2, y=\pm1, z=\pm1$. Use outward normals.
2. $\boldsymbol{F} = \langle 0,0,\cos(xy+2^z)\rangle$, S is the part of the cylinder $x^2+y^2=1$ with $0\le z\le 2$, using the usual parametrization and an outward normal.
3. $\boldsymbol{F} = \langle y+z, 2x+y, y\rangle$, S is the surface of the triangle with corners $(1,0,0)$, $(0,2,0)$, $(0,0,3)$ with normal pointing away from the origin.
4. $\boldsymbol{F} = \langle x,y,-z\rangle$, S is the surface of the ellipsoid $x^2+\frac{1}{4}y^2+\frac{1}{9}z^2 = 9$. Use the parametrization $\mathbf{\Phi}(\phi,\theta) = (3\sin\phi\cos\theta, 6\sin\phi\sin\theta, 9\cos\phi)$.
5. $\boldsymbol{F} = \langle 2y, 2x, z\rangle$, S is the part of the cone $x^2 = y^2+z^2$ inside the sphere $(x-8)^2+y^2+z^2 = 49$. Assume $\boldsymbol{n}\cdot\boldsymbol{i} < 0$.

6. $\boldsymbol{F} = \left\langle 3, x^2, y \right\rangle$, S is the part of the paraboloid $z = x^2 + y^2$ above the xy plane and below the plane $x + y + z = 9$. Assume $\boldsymbol{n} \cdot \boldsymbol{k} > 0$.

Answer: 1. You have to integrate over each of the six surfaces of the cube. This can be rather boring. In each case the normal will be in one of the three principal directions, which simplifies your integral. I get

$$\int_0^2 \int_{-1}^1 \left\langle x^2, y, -1 \right\rangle \cdot \langle 0, 0, -1 \rangle \; \mathrm{d}y \, \mathrm{d}x + \int_0^2 \int_{-1}^1 \left\langle x^2, y, 1 \right\rangle \cdot \langle 0, 0, 1 \rangle \; \mathrm{d}y \, \mathrm{d}x +$$
$$\int_0^2 \int_{-1}^1 \left\langle x^2, -1, z^3 \right\rangle \cdot \langle 0, -1, 0 \rangle \; \mathrm{d}z \, \mathrm{d}x + \int_0^2 \int_{-1}^1 \left\langle x^2, 1, z^3 \right\rangle \cdot \langle 0, 1, 0 \rangle \; \mathrm{d}z \, \mathrm{d}x +$$
$$\int_{-1}^1 \int_{-1}^1 \left\langle 0, y, z^3 \right\rangle \cdot \langle -1, 0, 0 \rangle \; \mathrm{d}z \, \mathrm{d}y + \int_{-1}^1 \int_{-1}^1 \left\langle 4, y, z^3 \right\rangle \cdot \langle 1, 0, 0 \rangle \; \mathrm{d}z \, \mathrm{d}y$$

The answer I get is 32.

2. The answer is 0; note the field is always orthogonal to the normal, so you are integrating 0, which is 0.

3. The integral is something like

$$\int_0^1 \int_0^{2-2x} y - 6x + 9 \; \mathrm{d}y \, \mathrm{d}x.$$

4. When we study the divergence theorem later, we will see this is the volume of the ellipsoid. I believe the answer is 216π.

5. This surface was considered in Problem 36. Use the parametrization

$$\boldsymbol{\Phi}(r, \theta) = \langle r, r\cos\theta, r\sin\theta \rangle, \quad \text{which gives} \quad \boldsymbol{T}_r \times \boldsymbol{T}_\theta = \langle r, -r\cos\theta, -r\sin\theta \rangle$$

This parametrization is orientation reversing because $\boldsymbol{T}_r \times \boldsymbol{T}_\theta \cdot \boldsymbol{i} > 0$. So we will use $\mathrm{d}\boldsymbol{S} = \boldsymbol{T}_\theta \times \boldsymbol{T}_r \, \mathrm{d}r \, \mathrm{d}\theta$, giving

$$\int_0^{2\pi} \int_{4-\sqrt{17/2}}^{4+\sqrt{17/2}} \langle 2r\cos\theta, 2r, r\sin\theta \rangle \cdot \langle -r, r\cos\theta, r\sin\theta \rangle \; \mathrm{d}r \, \mathrm{d}\theta = \frac{\pi}{3} \, r^3 \Big|_{4-\sqrt{17/2}}^{4+\sqrt{17/2}} = \frac{113\pi}{3} \sqrt{\frac{17}{2}}$$

6. This is kind of difficult. I made some substitutions and got

$$\iint_D \langle 3, u^2, v\rangle \cdot \langle -2u, -2v, 1\rangle \,\mathrm{d}u\,\mathrm{d}v = \iint_D -6u - 2u^2v + v \,\mathrm{d}u\,\mathrm{d}v$$

$$= \iint_{\tilde{D}} -6\left(\tilde{u} - \frac{1}{2}\right) - 2\left(\tilde{u} - \frac{1}{2}\right)^2\left(\tilde{v} - \frac{1}{2}\right) + \left(\tilde{v} - \frac{1}{2}\right) \mathrm{d}\tilde{u}\,\mathrm{d}\tilde{v}$$

$$= \int_0^{2\pi}\int_0^{\sqrt{19/2}} \left[\frac{11}{4} - 6r\cos\theta - 2r^3\sin\theta\cos^2\theta + 2r^2\sin\theta\cos\theta\right] r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= \frac{11}{2}\frac{19}{4}\pi - \frac{4}{15}\left(\frac{19}{2}\right)^{5/2}$$

YMMV.

39. (misc.) Let C be a curve representing the intersection of the sphere $x^2 + y^2 + z^2 = r^2$ and the plane $x + y + z = 0$. Let $f(x, y, z) = x^2$ be the mass density of a wire running along C. What is the total mass of the wire.

Answer: This is essentially #13 from §7.1. The hard part is getting the parametrization. I suggest you try to find one of the form $\boldsymbol{c}(\theta) = r\cos\theta\boldsymbol{u} + r\sin\theta\boldsymbol{v}$, where $\boldsymbol{u}, \boldsymbol{v}$ are unit vectors which are orthogonal to one another, and which are parallel to the plane, *i.e.*, which are normal to $\langle 1, 1, 1\rangle$. If you choose $\boldsymbol{u} = \left\langle 0, 1/\sqrt{2}, -1/\sqrt{2}\right\rangle$, and $\boldsymbol{v} = \left\langle -2/\sqrt{6}, 1/\sqrt{6}, 1/\sqrt{6}\right\rangle$, you will find that the x coordinate is $-(2r\sin\theta)/\sqrt{6}$. We will also have

$$\boldsymbol{c}'(\theta) = -r\sin\theta\boldsymbol{u} + r\cos\theta\boldsymbol{v},$$

the length of which is r. Thus our path integral is

$$\int_0^{2\pi} \frac{4}{6}r^2\sin^2\theta r\,\mathrm{d}\theta = \frac{2}{3}\pi r^3$$

40. (misc.) Let $\phi(x, y, z) = e^x + yz$ represent the electric potential in space. What is the work done by the field $-\nabla\phi$ on a particle which moves from $(0, 2, 3)$ to $(3, -2, 1)$.

Answer: This is a line integral. The answer should be

$$-\phi(x, y, z)\Big|_{(0,2,3)}^{(3,-2,1)} = -e^3 + 2 + 1 + 6 = 9 - e^3$$

Final Exam Prep Sheet:

41. (§8.1) State Green's Theorem.

Answer: If D is a region in the plane with boundary C, then for $\mathcal{C}^1$ functions P, Q

$$\int_{C^+} P\,\mathrm{d}x + Q\,\mathrm{d}y = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y,$$

where C^+ is the positive orientation of curve C. Note that the integral on the left hand side is a line integral (sometimes called a "circulation" because it is a line integral over a closed curve).

42. (§8.1) Given some region in the plane, D, bounded by the closed curve, C, find the area of D.

1. C is traced out by $\boldsymbol{c}(t) = \left\langle \sin t \cos t, \sin^2 t, 0 \right\rangle$ for $0 \le t \le \pi$. (From last year's final)
2. C is traced out by $r = r(\theta)$ for $0 \le \theta \le 2\pi$. That is, the curve C is given by $\boldsymbol{c}(t) = \langle r(t)\cos t, r(t)\sin t, 0\rangle$ for $0 \le t \le 2\pi$. (Note that in order for C to be a closed curve we need to have $r(0) = r(2\pi)$, and $r(t) \ge 0$)
3. C is traced out by $r = r(\theta) = 2\pi\theta - \theta^2$ for $0 \le \theta \le 2\pi$.
4. Let C be traced out by $\boldsymbol{c}(t) = \left\langle \cos^2 t, \cos t \sin t, 0 \right\rangle$, for $-\pi/2 \le t \le \pi/2$. (Oddly enough, this problem doesn't really require any calculus.)
5. Let C be traced out by $\boldsymbol{c}(t) = \left\langle \cos^3 t, \cos^2 t \sin t, 0 \right\rangle$, for $-\pi/2 \le t \le \pi/2$.
6. Let C be one loop of the lemniscate parametrized by $\boldsymbol{R}(t) = \left\langle \sqrt{\cos 2t}\cos t, \sqrt{\cos 2t}\sin t, 0 \right\rangle$ for $-\pi/4 \le t \le \pi/4$.

Answer: Because this question is labelled §8.1, you can guess that Green's Theorem will be used. On the exam, I will not give you such a hint, so you should be able to recognize when Green's Theorem applies, and be able to apply it.

For this problem we can pick *any* field $\boldsymbol{F} = \langle P, Q, 0\rangle$ such that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$, then

$$\int_{\boldsymbol{c}} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s}$$

gives the area of D.

Popular choices of $\boldsymbol{F}$ include $\boldsymbol{F}_1 = \langle 0, x, 0\rangle$, $\boldsymbol{F}_2 = \langle -y, 0, 0\rangle$, and $\boldsymbol{F}_3 = \frac{1}{2}\langle -y, x, 0\rangle$. You could pick a more exotic $\boldsymbol{F}$, but this is not to your advantage, since you will have to integrate this field.

1. Let's try $\boldsymbol{F} = \langle 0, x, 0\rangle$. Then we have to find

$$\int_0^{\pi} \langle 0, \sin t\cos t, 0\rangle \cdot \langle \ldots, 2\sin t \cos t, 0\rangle \; \mathrm{d}t = \int_0^{\pi} 2\sin^2 t \cos^2 t \,\mathrm{d}t = \frac{1}{2}\int_0^{\pi} (2\sin t\cos t)^2 \; \mathrm{d}t$$

$$= \frac{1}{2}\int_0^{\pi} \sin^2 2t \,\mathrm{d}t = \frac{1}{2}\int_0^{\pi} \frac{1}{2}(1 - \cos 4t) \; \mathrm{d}t = \frac{1}{4} \; (t - \sin 4t)\Big|_0^{\pi} = \frac{\pi}{4}$$

(From last year's final)

2. Using $\boldsymbol{F} = \frac{1}{2}\langle -y, x, 0\rangle$, we have

$$\frac{1}{2}\int_0^{2\pi} \langle -r(t)\sin t, r(t)\cos t, 0\rangle \cdot \left\langle r'(t)\cos t - r(t)\sin t, r'(t)\sin t + r(t)\cos t, 0 \right\rangle \, \mathrm{d}t$$

$$= \frac{1}{2}\int_0^{2\pi} -r(t)r'(t)\sin t\cos t + r^2(t)\sin^2 t + r(t)r'(t)\sin t\cos t + r^2(t)\cos^2 t \,\mathrm{d}t$$

$$= \frac{1}{2}\int_0^{2\pi} r^2(t)\,\mathrm{d}t$$

3. Using the previous problem this would become

$$\frac{1}{2}\int_0^{2\pi} \left(2\pi t - t^2\right)^2 \, \mathrm{d}t = \frac{1}{2}\int_0^{2\pi} 4\pi^2 t^2 + t^4 - 4\pi t^3 \, \mathrm{d}t = \frac{1}{2} \left(4\pi^2 t^3/3 + t^5/5 - \pi t^4\right)\Big|_0^{2\pi} = 32\pi^5/30.$$

4. Using $\boldsymbol{F} = \frac{1}{2}\langle -y, x, 0\rangle$, you eventually get

$$\frac{1}{2}\int_{-\pi/2}^{\pi/2} \cos^2 t \, \mathrm{d}t = \frac{\pi}{4}.$$

The slick way to do this is to rewrite $\boldsymbol{c}(t) = \left\langle \frac{1}{2} + \frac{1}{2}\cos 2t, \frac{1}{2}\sin 2t, 0\right\rangle$, for $-\pi/2 \leq t \leq \pi/2$, which you can reparametrize by $\tilde{\boldsymbol{c}}(t) = \left\langle \frac{1}{2} + \frac{1}{2}\cos t, \frac{1}{2}\sin t, 0\right\rangle$, for $-\pi \leq t \leq \pi$, which is a circle of radius $\frac{1}{2}$ centered at $\left(\frac{1}{2}, 0, 0\right)$. This gives the right answer.

5. Using $\boldsymbol{F} = \frac{1}{2}\langle -y, x, 0\rangle$, you eventually get

$$\frac{1}{2}\int_{-\pi/2}^{\pi/2} \cos^4 t \, \mathrm{d}t = \frac{1}{2}\int_{-\pi/2}^{\pi/2} \left(\frac{1}{2} + \frac{1}{2}\cos 2t\right)^2 \, \mathrm{d}t = \frac{3\pi}{16}.$$

6. I think the answer is $\frac{1}{2}$.

43. (§8.2) State Stokes' Theorem.

Answer: If S is an oriented surface with oriented boundary ∂S, and $\boldsymbol{F}$ is a $\mathcal{C}^1$ vector field, then

$$\int_{\partial S} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = \iint_S (\nabla \times \boldsymbol{F}) \cdot \mathrm{d}\boldsymbol{S}$$

44. (§8.2) Given some surface, S, and a vector field $\boldsymbol{F}$ find the surface integral of the curl:

$$\iint_S (\nabla \times \boldsymbol{F}) \cdot \mathrm{d}\boldsymbol{S}.$$

1. $\boldsymbol{F} = \langle -z - y, x + y, z + x\rangle$, and S is the hemisphere $x^2 + y^2 + z^2 = 1$, with $z \geq 0$, oriented such that $\boldsymbol{n} \cdot \boldsymbol{k} \geq 0$. (From last year's final)
2. $\boldsymbol{F} = \left\langle x^2 + y - 4, 3xy, 2xz + z^2\right\rangle$, S is the surface of the hemisphere $x^2 + y^2 + z^2 = 16$ above the xy plane.
3. $\boldsymbol{F} = \left\langle x \cos y e^z, x^2 - y^2, \arctan z + y^x\right\rangle$, and S is the (closed) surface given by $16x^2 + 9y^2 + 4z^2 = 1$.
4. $\boldsymbol{F} = \left\langle y^2, z^2, x^2\right\rangle$, and S is the hemisphere $x^2 + y^2 + z^2 = 9$, with $z \geq 0$ oriented such that $\boldsymbol{n} \cdot \boldsymbol{k} \geq 0$.
5. $\boldsymbol{F} = \langle -y, x, 2x + z\rangle$, and S is the surface $\left\{(x, y, z) \,\middle|\, z^2 = x^2 + y^2,\ 2 \leq z \leq 3\right\}$ oriented such that $\boldsymbol{n} \cdot \boldsymbol{k} \leq 0$.

Answer: Because this question is labelled §8.2, you can guess that Stokes' Theorem will be used. On the exam, I will not give you such a hint, so you should be prepared to answer questions of this type directly when necessary, and you should be able to recognize when Stokes' Theorem applies, and be able to apply it.

1. With ∂S the circle $x^2 + y^2 = 1$, and $z = 0$, we have

$$\int_{\partial S} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = \int_0^{2\pi} \langle 0 - y, x + y, 0 + x \rangle \cdot \langle -y, x, 0 \rangle \; \mathrm{d}t = \int_0^{2\pi} y^2 + x^2 + xy \, \mathrm{d}t$$

$$= \int_0^{2\pi} 1 + \cos t \sin t \, \mathrm{d}t = t + \left. \frac{\sin^2 t}{2} \right|_0^{2\pi} . = 2\pi.$$

Here we have used the fact that for the parametrization $\langle x, y, z \rangle = \boldsymbol{c}(t) = \langle \cos t, \sin t, 0 \rangle$ that $\mathrm{d}x = -\sin t \, \mathrm{d}t = -y \, \mathrm{d}t$ and similarly $\mathrm{d}y = x \, \mathrm{d}t$. (From last year's final)

2. Use Stokes' Theorem *twice* to get this equivalent to integrating the flux of $\boldsymbol{F}$ over S', which is the disc $x^2 + y^2 \le 16$ in the xy plane. Thus you only have to concern yourself with the $\boldsymbol{k}$ component of $\boldsymbol{F}$, which is $3y - 1$. Over the disc the $3y$ part disappears due to symmetry, and we are left with the integral of -1 over S', which is negative the area of S', *i.e.*, -16π.
Or you could use Stokes' Theorem only once to get the line integral over the circle itself:

$$\int_{\partial S} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = \int_0^{2\pi} \left\langle x^2 + y - 4, 3xy, 0 + 0 \right\rangle \cdot \langle -y, x, 0 \rangle \; \mathrm{d}t = \int_0^{} -yx^2 - y^2 + 4y + 3x^2 y \, \mathrm{d}t$$

$$= \int_0^{2\pi} -y^2 + 4y + 2x^2 y \, \mathrm{d}t = \int_0^{2\pi} -16 \sin^2 t + 16 \sin t + 128 \cos^2 t \sin t \, \mathrm{d}t$$

$$= \; -8t + 8 \cos 2t - 16 \cos t + (128/3) \cos^3 t \Big|_0^{2\pi} = -16\pi.$$

3. The flux of $\nabla \times \boldsymbol{F}$ for *any* $\boldsymbol{F}$ over *any* closed surface is 0.
4. If you use Stokes' Theorem once, you will get

$$\int_0^{2\pi} \left\langle 9 \sin^2 t, 0, 9 \cos^2 t \right\rangle \cdot \langle -3 \sin t, 3 \cos t, 0 \rangle \; \mathrm{d}t = \int_0^{2\pi} -27 \sin^3 t \, \mathrm{d}t$$

$$= -27 \int_0^{2\pi} \sin t \left(1 - \cos^2 t\right) \; \mathrm{d}t = 0$$

The alternative is to use Stokes' Theorem twice: the flux of the curl is equivalent to

$$\iint_{\partial S_1} (\nabla \times \boldsymbol{F}) \cdot \mathrm{d}\boldsymbol{S}$$

If we choose S_1 to be the disc $x^2 + y^2 \leq 9$, $z = 0$ we have $\mathrm{d}\boldsymbol{S} = \boldsymbol{k}\,\mathrm{d}A = \langle 0, 0, 1\rangle\ \mathrm{d}A$, and $\nabla \times \boldsymbol{F} = \langle -2z, -2x, -2y\rangle$. So our answer is

$$\iint\limits_{\partial S_1} -2y\,\mathrm{d}A = 0$$

because the region is symmetric in y.

5. Use Stokes' Theorem once; the boundary of S has two components, parametrized by $\boldsymbol{c_1}(t) = \langle 2\cos t, 2\sin t, 2\rangle$, and $\boldsymbol{c_2}(t) = \langle 3\cos t, -3\sin t, 3\rangle$. Our answer is then

$$\int_0^{2\pi} \langle -2\sin t, 2\cos t, 2\rangle \cdot \langle -2\sin t, 2\cos t, 0\rangle\ \mathrm{d}t + \int_0^{2\pi} \langle 3\sin t, 3\cos t, 3\rangle \cdot \langle -3\sin t, -3\cos t, 0\rangle\ \mathrm{d}t$$

$$= \int_0^{2\pi} -5\,\mathrm{d}t = -10\pi$$

45. (§8.2) Given some closed curve C parametrized by $\boldsymbol{c}(t)$, and a vector field $\boldsymbol{F}$, find the circulation of the vector field:

$$\int\limits_{\boldsymbol{c}} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s}.$$

1. $\boldsymbol{F} = \boldsymbol{r} = \langle x, y, z\rangle$, with $\boldsymbol{c}(t) = \langle 2\cos t, 2\sin t, 0\rangle$, for $0 \leq t \leq 2\pi$.
2. $\boldsymbol{F} = \langle 6xz, 2xz, 0\rangle$, with C the boundary of triangle going from $(1, 0, 0)$ to $(0, 3, 0)$ to $(0, 0, 2)$.

Answer: Because this question is labelled §8.2, you can guess that Stokes' Theorem will be used; that is, you should evaluate this as

$$\iint\limits_{S} (\nabla \times \boldsymbol{F}) \cdot \mathrm{d}\boldsymbol{S},$$

where S is some oriented surface that has the curve traced out by $\boldsymbol{c}$ as its boundary.

1. Because $\boldsymbol{F} = \langle x, y, z\rangle$, we have $\nabla \times \boldsymbol{F} = \boldsymbol{0}$, so whichever surface S we pick, we get $\iint_S (\nabla \times \boldsymbol{F}) \cdot \mathrm{d}\boldsymbol{S} = 0$. This is not the best way to do this problem, because this field is conservative: $\boldsymbol{F} = \nabla\phi$, with $\phi = \frac{1}{2}\left(x^2 + y^2 + z^2\right)$. You should remember that the circulation of a conservative field is always zero. (See Problem 46.)
2. $\nabla \times \boldsymbol{F} = \langle -2x, 6x, 2z\rangle$. The surface S with C as oriented boundary is the plane containing the three points $(1, 0, 0)$, $(0, 3, 0)$, $(0, 0, 2)$, that is the plane with equation $6x + 2y + 3z - 6 = 0$. Use the graph parametrization $\boldsymbol{\Phi}(u, v) = \langle u, v, 2 - 2u - 2v/3\rangle$.

Then $\boldsymbol{T}_u \times \boldsymbol{T}_v = \langle 2, 2/3, 1\rangle$. This gives the surface integral

$$\int_0^1 \int_0^{3-3u} \langle -2u, 6u, 4-4u-4v/3\rangle \cdot \langle 2, 2/3, 1\rangle \,\mathrm{d}v\,\mathrm{d}u = \int_0^1 \int_0^{3-3u} 4-4u-4v/3 \,\mathrm{d}v\,\mathrm{d}u$$

$$= \int_0^1 (4-4u)\,v - 2v^2/3 \Big|_0^{3-3u} \mathrm{d}u = \int_0^1 (4-4u)(3-3u) - \frac{2}{3}(3-3u)^2 \,\mathrm{d}u$$

$$= \int_0^1 (4-4u)(3-3u) - (2-2u)(3-3u)\,\mathrm{d}u = \int_0^1 (2-2u)(3-3u)\,\mathrm{d}u$$

$$= \int_0^1 6 - 12u + 6u^2 \,\mathrm{d}u = 6 - 6 + 2 = 2$$

Doing it this way seems a little less annoying than doing the line integral the long way.

46. (§8.3) Define what it means for a field, $\boldsymbol{F}$, defined over all of $\mathbb{R}^3$, to be conservative. Give some equivalent conditions.

Answer: $\boldsymbol{F}$ is said to be conservative if $\boldsymbol{F} = \nabla\phi$ for some scalar field ϕ. This is equiavalent to the following:

1. For every closed curve, C, the circulation $\int_C \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{s} = 0$.
2. For any pair of curves C_1, C_2 with the same start and endpoints, $\int_{C_1} \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{s} = \int_{C_2} \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{s}$
3. $\nabla\times\boldsymbol{F} = \boldsymbol{0}$

47. (§8.3) Given a field, $\boldsymbol{F}$, determine if $\boldsymbol{F}$ is conservative, and if it is, find a potential for it, *i.e.*, find a ϕ such that $\boldsymbol{F} = \nabla\phi$.

1. $\boldsymbol{F} = \left\langle \cos x + 2yx, x^2, e^z \right\rangle$.
2. $\boldsymbol{F} = \langle xyz, xyz, xyz\rangle$.
3. $\boldsymbol{F} = \langle -y, x, 0\rangle$.

Answer: An easy way to do this is to check if $\nabla\times\boldsymbol{F} = \boldsymbol{0}$. Alternatively, you could *assume* $\boldsymbol{F}$ is conservative, and integrate three times to try to find a potential for the field. The latter strategy is probably better, as the former strategy can give false positives if the field is not defined on an infinite number of points. See also Problem 34.

1. Conservative, with $\phi = \sin x + x^2 y + e^z + C$.
2. Not conservative, $\nabla\times\boldsymbol{F} = \langle xz - xy, xy - yz, yz - xz\rangle \neq \boldsymbol{0}$. Note that the curl test is ok in this case because it gave us an answer "no." We only have to beware the curl test if it tells us "yes, this field is conservative."
3. Not conservative, $\nabla\times\boldsymbol{F} = \langle 0, 0, 2\rangle \neq \boldsymbol{0}$.

48. (§8.3) Given a field, $\boldsymbol{F}$, and a curve, C, find the line integral

$$\int_C \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{S}$$

1. $\boldsymbol{F} = \left\langle 5x^4 - 2xy^3, -3x^2y^2, 2z \right\rangle$, and C is the straight line from $(0,0,0)$ to $(2,-2,1)$.

Answer: Because this is associated with §8.3, you should guess that something about conservative fields is relevant.

1. In this case $\boldsymbol{F} = \nabla\phi$ for $\phi = x^5 - x^2y^3 + z^2$. So the answer is

$$x^5 - x^2y^3 + z^2\Big|_{(0,0,0)}^{(2,-2,1)} = (32 + 32 + 1) - (0 + 0 + 0) = 65$$

49. (§8.4) State the Divergence Theorem.

Answer: If W is a region in $\mathbb{R}^3$ with oriented boundary ∂W, and $\boldsymbol{F}$ is a $\mathcal{C}^1$ vector field, then

$$\iint_{\partial W} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} = \iiint_{W} \nabla \cdot \boldsymbol{F}\,\mathrm{d}V$$

50. (§8.4) Given some surface, S, and a vector field $\boldsymbol{F}$ find the integral of the flux:

$$\iint_{S} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S}.$$

1. $\boldsymbol{F} = \left\langle \frac{y}{b^2}, \frac{-x}{a^2}, 1 \right\rangle$, S is the ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$. Use outward pointing normals.
2. $\boldsymbol{F} = \boldsymbol{r} = \langle x, y, z \rangle$, S is the surface of some set W with volume V. Use outward pointing normals.
3. $\boldsymbol{F} = \langle x + z, x + y, \cos(x) - z \rangle$, S is the surface of some set W with volume V. Use outward pointing normals.
4. $\boldsymbol{F} = \langle x^2, y^2, z^3 \rangle$, S is the surface of the cube bounded by $x = \pm 1, y = \pm 1, z = \pm 1$. Use outward pointing normals.
5. $\boldsymbol{F} = \langle x^2, y, z^3 \rangle$, S is the surface of the cube bounded by $x = 0, x = 2, y = \pm 1, z = \pm 1$. Use outward normals.
6. $\boldsymbol{F} = \langle 4x, -2y^2, z^2 \rangle$, S is the cylinder $\left\{(x, y, z) \,\middle|\, x^2 + y^2 = 4,\ -3 \le z \le 3\right\}$. Assume the normal points away from the z axis.
7. $\boldsymbol{F} = \left\langle \frac{1}{4}x, z, x - \frac{y}{2} \right\rangle$, and S is the boundary of the region W bounded by the inequalities $\frac{1}{16}x^2 + \frac{1}{4}y^2 + z^2 \le 1$, and $z \ge 0$. Assume outward normals. (Note in this case we are taking S to be the closed surface which is the boundary of W.)

Answer: Because this question is labelled §8.4, you can guess that the divergence theorem will be used. On the exam, I will not give you such a hint, so you should be prepared to answer questions of this type directly when necessary, and you should be able to recognize when the divergence theorem applies, and be able to apply it.

1. Note that S is a closed surface, and $\nabla \cdot \boldsymbol{F} = 0$ everywhere, so the answer is $\iiint_W 0\,\mathrm{d}V = 0$.
2. Because W is given so ambiguously, we *have* to use the Divergence Theorem; S is a closed surface, and $\nabla \cdot \boldsymbol{F} = 3$ everywhere, so the answer is $3V$.
3. Because W is given so ambiguously, we *have* to use the Divergence Theorem; S is a closed surface, and $\nabla \cdot \boldsymbol{F} = 1$ everywhere, so the answer is $1V$.

4. By the divergence theorem, this is

$$\iiint_W 2x + 2y + 3z^2 \,\mathrm{d}V$$

over the cube W. By symmetry the first two parts of the integrand integrate to 0 over the cube, so this is $\iiint_W 3z^2 \,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x = \iint 2\,\mathrm{d}y\,\mathrm{d}x = 8$.

5. This appeared in Problem 38. With $\nabla \cdot \boldsymbol{F} = 2x + 1 + 3z^2$, the answer is

$$\iiint_W 2x + 1 + 3z^2 \,\mathrm{d}V = 8 + 16 + 8 = 32$$

6. Note that this surface S is *not* a closed surface. It would seem that the divergence theorem would not apply. However, it does, but in a weird way: let S_t be the surface $\{(x, y, z) \mid x^2 + y^2 \le 4,\ z = 3\}$, and let S_b be the surface $\{(x, y, z) \mid x^2 + y^2 \le 4,\ z = -3\}$. Then $\Sigma = S \cup S_t \cup S_b$ *is* a closed surface. We essentially added the 'top' and 'bottom' of a can to get the closed can. By divergence theorem, we have

$$\iiint_W \nabla \cdot \boldsymbol{F} \,\mathrm{d}W = \iint_S \boldsymbol{F} \cdot \,\mathrm{d}\boldsymbol{S} + \iint_{S_t} \boldsymbol{F} \cdot \,\mathrm{d}\boldsymbol{S} + \iint_{S_b} \boldsymbol{F} \cdot \,\mathrm{d}\boldsymbol{S}$$

Now something funny happens:

$$\iint_{S_t} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} = \iint \langle 4x, -2y^2, 9\rangle \cdot \langle 0, 0, 1\rangle \,\mathrm{d}S = -\iint \langle 4x, -2y^2, 9\rangle \cdot \langle 0, 0, -1\rangle \,\mathrm{d}S = -\iint_{S_b} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S}$$

So, in the end,

$$\iiint_W \nabla \cdot \boldsymbol{F} \,\mathrm{d}W = \iint_S \boldsymbol{F} \cdot \,\mathrm{d}\boldsymbol{S} + 0$$

Now note that $\nabla \cdot \boldsymbol{F} = 4 - 4y + 2z$. By the symmetry of W we have

$$\iiint_W 4 - 4y + 2z \,\mathrm{d}W = \iiint_W 4 \,\mathrm{d}W = 4\mathrm{Vol}(W) = 4\,(24\pi) = 96\pi$$

7. First note that $\nabla \cdot \boldsymbol{F} = \frac{1}{4} + 0 + 0$. Thus we are trying to find one quarter the volume of W. To find $\frac{1}{4} \iiint_W \,\mathrm{d}V$, we make the change of variables: $x = 4u$, $y = 2v$, $z = w$. Then we have

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \det \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 8$$

So we have

$$\frac{1}{4} \iiint_W \,\mathrm{d}V = \frac{1}{4} \iiint_{\tilde{W}} |8| \,\mathrm{d}\tilde{V},$$

where, by direct substitution of our change of variables into the inequalities $\frac{1}{16}x^2 + \frac{1}{4}y^2 + z^2 \leq 1$, and $z \geq 0$, $\tilde{W}$ is the region bounded by $u^2 + v^2 + w^2 \leq 1$, and $w \geq 0$. Then $\iiint_{\tilde{W}} \mathrm{d}\tilde{V} = 2/3\pi$, the volume of a hemisphere of radius 1. Thus the answer to the problem is $(8/4)(2/3)\pi = 4\pi/3$.

51. (misc.) Let ϕ be a differentiable scalar field, let $\boldsymbol{F} = \nabla\phi / \|\nabla\phi\|$ be a vector field, and let W be the region in $\mathbb{R}^3$ defined by $\{(x, y, z) \mid \phi(x, y, z) \leq 17\}$. Find $\iiint_W \nabla \cdot \boldsymbol{F} \, \mathrm{d}V$.

Answer: This seems like a clear use of divergence theorem. We have

$$\iiint\limits_W \nabla \cdot \boldsymbol{F} \, \mathrm{d}V = \iint\limits_S \boldsymbol{F} \cdot \, \mathrm{d}\boldsymbol{S} = \iint\limits_S \boldsymbol{F} \cdot \boldsymbol{n} \, \mathrm{d}S,$$

where S is the boundary of the set W *i.e.*, S is $\{(x, y, z) \mid \phi(x, y, z) = 17\}$, and $\boldsymbol{n}$ is the outward pointing unit normal. Since S is a level set of ϕ, $\nabla\phi$ is normal to S. Then $\boldsymbol{F}$ is a unit length vector field normal to S (that is, at any point P on S, $\boldsymbol{F}$ evaluated at P is a unit vector normal to S at P). As $\boldsymbol{F}$ is in the direction of increase of ϕ, it is pointing towards the outside of W. That is $\boldsymbol{F} = \boldsymbol{n}$, so we are trying to find

$$\iint\limits_S \boldsymbol{F} \cdot \boldsymbol{n} \, \mathrm{d}S = \iint\limits_S \boldsymbol{n} \cdot \boldsymbol{n} \, \mathrm{d}S = \iint\limits_S 1 \, \mathrm{d}S$$

thus the answer is the area of S.

52. (misc.) Let $\boldsymbol{c}(t) = \left(e^{2t}/\sqrt{3}\right)\langle 1, 1, 1\rangle$ be the path of a particle. Show that the particle is always moving in the direction of maximal increase of the scalar function $\phi(x, y, z) = x^2 + y^2 + z^2$.

Answer: First calculate the velocity vector: $\boldsymbol{c}'(t) = \left(2e^{2t}/\sqrt{3}\right)\langle 1, 1, 1\rangle = 2\boldsymbol{c}(t)$. Now note that the direction of maximal increase of ϕ at a point (x, y, z) is $\nabla\phi(x, y, z) = \langle 2x, 2y, 2z\rangle$. Thus $\nabla\phi(\boldsymbol{c}(t)) = 2\boldsymbol{c}(t) = \boldsymbol{c}'(t)$, and thus the particle is moving in the direction of maximal increase.