Final Exam Preparation

The Final Exam is Wednesday, March 16th, and is comprehensive, covering all the material we have looked at this quarter, but with a particular emphasis on chapters §8.1–8.4. *Please bring a blue book for the exam.* You may not use a calculator or notes. The following formulæ will be provided on your exam:

$$\nabla \quad \text{abbreviates} \quad \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z} \qquad \mathrm{d}\boldsymbol{s} = \boldsymbol{c}'(t)\,\mathrm{d}t$$

$$\mathrm{d}\boldsymbol{S} = \boldsymbol{T}_u \times \boldsymbol{T}_v \,\mathrm{d}u\,\mathrm{d}v = \left(\frac{\partial \boldsymbol{\Phi}}{\partial u} \times \frac{\partial \boldsymbol{\Phi}}{\partial v}\right)\mathrm{d}u\,\mathrm{d}v \qquad \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

Everything else must be committed to memory.

What follows are questions similar to your homework assignments. You should be prepared to answer at least questions like these. This list may not be exhaustive.

Final Exam Prep Sheet:

1. (§8.1) State Green's Theorem.

2. (§8.1) Given some region in the plane, D, bounded by the closed curve, C, find the area of D.

1. C is traced out by $\boldsymbol{c}(t) = \left\langle \sin t \cos t, \sin^2 t, 0 \right\rangle$ for $0 \le t \le \pi$. (From last year's final)
2. C is traced out by $r = r(\theta)$ for $0 \le \theta \le 2\pi$. That is, the curve C is given by $\boldsymbol{c}(t) = \langle r(t)\cos t, r(t)\sin t, 0\rangle$ for $0 \le t \le 2\pi$. (Note that in order for C to be a closed curve we need to have $r(0) = r(2\pi)$, and $r(t) \ge 0$)
3. C is traced out by $r = r(\theta) = 2\pi\theta - \theta^2$ for $0 \le \theta \le 2\pi$.
4. Let C be traced out by $\boldsymbol{c}(t) = \left\langle \cos^2 t, \cos t \sin t, 0 \right\rangle$, for $-\pi/2 \le t \le \pi/2$. (Oddly enough, this problem doesn't really require any calculus.)
5. Let C be traced out by $\boldsymbol{c}(t) = \left\langle \cos^3 t, \cos^2 t \sin t, 0 \right\rangle$, for $-\pi/2 \le t \le \pi/2$.
6. Let C be one loop of the lemniscate parametrized by $\boldsymbol{R}(t) = \left\langle \sqrt{\cos 2t}\cos t, \sqrt{\cos 2t}\sin t, 0 \right\rangle$ for $-\pi/4 \le t \le \pi/4$.

3. (§8.2) State Stokes' Theorem.

4. (§8.2) Given some surface, S, and a vector field $\boldsymbol{F}$ find the surface integral of the curl:

$$\iint_S (\nabla \times \boldsymbol{F}) \cdot \mathrm{d}\boldsymbol{S}.$$

1. $\boldsymbol{F} = \langle -z - y, x + y, z + x \rangle$, and S is the hemisphere $x^2 + y^2 + z^2 = 1$, with $z \ge 0$, oriented such that $\boldsymbol{n} \cdot \boldsymbol{k} \ge 0$. (From last year's final)
2. $\boldsymbol{F} = \left\langle x^2 + y - 4, 3xy, 2xz + z^2 \right\rangle$, S is the surface of the hemisphere $x^2 + y^2 + z^2 = 16$ above the xy plane.
3. $\boldsymbol{F} = \left\langle x \cos y e^z, x^2 - y^2, \arctan z + y^x \right\rangle$, and S is the (closed) surface given by $16x^2 + 9y^2 + 4z^2 = 1$.
4. $\boldsymbol{F} = \left\langle y^2, z^2, x^2 \right\rangle$, and S is the hemisphere $x^2 + y^2 + z^2 = 9$, with $z \ge 0$ oriented such that $\boldsymbol{n} \cdot \boldsymbol{k} \ge 0$.

5. $\boldsymbol{F} = \langle -y, x, 2x+z \rangle$, and S is the surface $\left\{(x,y,z) \,\middle|\, z^2 = x^2+y^2,\ 2 \le z \le 3\right\}$ oriented such that $\boldsymbol{n} \cdot \boldsymbol{k} \le 0$.

5. (§8.2) Given some closed curve C parametrized by $\boldsymbol{c}(t)$, and a vector field $\boldsymbol{F}$, find the circulation of the vector field:

$$\int_{\boldsymbol{c}} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s}.$$

1. $\boldsymbol{F} = \boldsymbol{r} = \langle x, y, z \rangle$, with $\boldsymbol{c}(t) = \langle 2\cos t, 2\sin t, 0 \rangle$, for $0 \le t \le 2\pi$.
2. $\boldsymbol{F} = \langle 6xz, 2xz, 0 \rangle$, with C the boundary of triangle going from $(1,0,0)$ to $(0,3,0)$ to $(0,0,2)$.

6. (§8.3) Define what it means for a field, $\boldsymbol{F}$, defined over all of $\mathbb{R}^3$, to be conservative. Give some equivalent conditions.

7. (§8.3) Given a field, $\boldsymbol{F}$, determine if $\boldsymbol{F}$ is conservative, and if it is, find a potential for it, *i.e.*, find a ϕ such that $\boldsymbol{F} = \nabla\phi$.

1. $\boldsymbol{F} = \left\langle \cos x + 2yx, x^2, e^z \right\rangle$.
2. $\boldsymbol{F} = \langle xyz, xyz, xyz \rangle$.
3. $\boldsymbol{F} = \langle -y, x, 0 \rangle$.

8. (§8.3) Given a field, $\boldsymbol{F}$, and a curve, C, find the line integral

$$\int_{C} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S}$$

1. $\boldsymbol{F} = \left\langle 5x^4 - 2xy^3, -3x^2y^2, 2z \right\rangle$, and C is the straight line from $(0,0,0)$ to $(2,-2,1)$.

9. (§8.4) State the Divergence Theorem.

10. (§8.4) Given some surface, S, and a vector field $\boldsymbol{F}$ find the integral of the flux:

$$\iint_{S} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S}.$$

1. $\boldsymbol{F} = \left\langle \frac{y}{b^2}, \frac{-x}{a^2}, 1 \right\rangle$, S is the ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$. Use outward pointing normals.
2. $\boldsymbol{F} = \boldsymbol{r} = \langle x, y, z \rangle$, S is the surface of some set W with volume V. Use outward pointing normals.
3. $\boldsymbol{F} = \langle x+z, x+y, \cos(x) - z \rangle$, S is the surface of some set W with volume V. Use outward pointing normals.
4. $\boldsymbol{F} = \left\langle x^2, y^2, z^3 \right\rangle$, S is the surface of the cube bounded by $x = \pm 1, y = \pm 1, z = \pm 1$. Use outward pointing normals.
5. $\boldsymbol{F} = \left\langle x^2, y, z^3 \right\rangle$, S is the surface of the cube bounded by $x = 0, x = 2, y = \pm 1, z = \pm 1$. Use outward normals.
6. $\boldsymbol{F} = \left\langle 4x, -2y^2, z^2 \right\rangle$, S is the cylinder $\left\{(x,y,z) \,\middle|\, x^2+y^2 = 4,\ -3 \le z \le 3\right\}$. Assume the normal points away from the z axis.

7. $\boldsymbol{F} = \left\langle \frac{1}{4}x, z, x - \frac{y}{2} \right\rangle$, and S is the boundary of the region W bounded by the inequalities $\frac{1}{16}x^2 + \frac{1}{4}y^2 + z^2 \leq 1$, and $z \geq 0$. Assume outward normals. (Note in this case we are taking S to be the closed surface which is the boundary of W.)

11. (misc.) Let ϕ be a differentiable scalar field, let $\boldsymbol{F} = \nabla\phi / \|\nabla\phi\|$ be a vector field, and let W be the region in $\mathbb{R}^3$ defined by $\{(x, y, z) \mid \phi(x, y, z) \leq 17\}$. Find $\iiint_W \nabla \cdot \boldsymbol{F}\, \mathrm{d}V$.

12. (misc.) Let $\boldsymbol{c}(t) = \left(e^{2t}/\sqrt{3}\right) \langle 1, 1, 1 \rangle$ be the path of a particle. Show that the particle is always moving in the direction of maximal increase of the scalar function $\phi(x, y, z) = x^2 + y^2 + z^2$.