A Delaunay-Laguerre Duality

Steven E. Pav
Department of Mathematics
Carnegie Mellon University
Pittsburgh, PA
spav++@andrew.cmu.edu

Carnegie Mellon

Abstract

The Laguerre Diagram generalizes Voronoi Diagrams to hyperball sites under the power distance. A well-known *application* of the Laguerre Diagram is to determine, given a set of hyperballs in $\mathbb{R}^n$, and a point, if the point is inside any of the hyperballs. The structure of the Laguerre Diagram, given certain conditions regarding the sites, may be of some use to the meshing *theoretician* in establishing lemmata regarding Delaunay Triangulations. We outline here a simple, though apparently unknown, duality between Delaunay Triangulations and the Laguerre Diagram. Briefly, any simplex from a collection of simplices with the Delaunay property is contained in the Laguerre cell of its circumball, with respect to the collection of circumballs of the simplices.

1. Introduction

The Laguerre Diagram is a well-known generalization of the Voronoi Diagram, but with spheres serving as sites, and using the "power distance" instead of the Euclidian distance. Formally we define the power of a point $x \in \mathbb{R}^n$ with respect to a ball, B with center c and radius r as

$$\mathcal{P}(B, x) = \mathrm{d}(x, c)^2 - r^2$$

Given a set of balls, Ω, for $\mathcal{B} \in \Omega$, the Laguerre cell of $\mathcal{B}$ is defined by

$$\mathrm{Dom}_{\Omega}(\mathcal{B}) = \left\{x \in \mathbb{R}^n \mid \mathcal{P}(\mathcal{B}, x) \leq \mathcal{P}\left(\mathcal{B}', x\right), \forall \mathcal{B}' \in \Omega\right\}.$$

The Laguerre Diagram is the collection of boundaries of the Laguerre cells of balls in Ω.

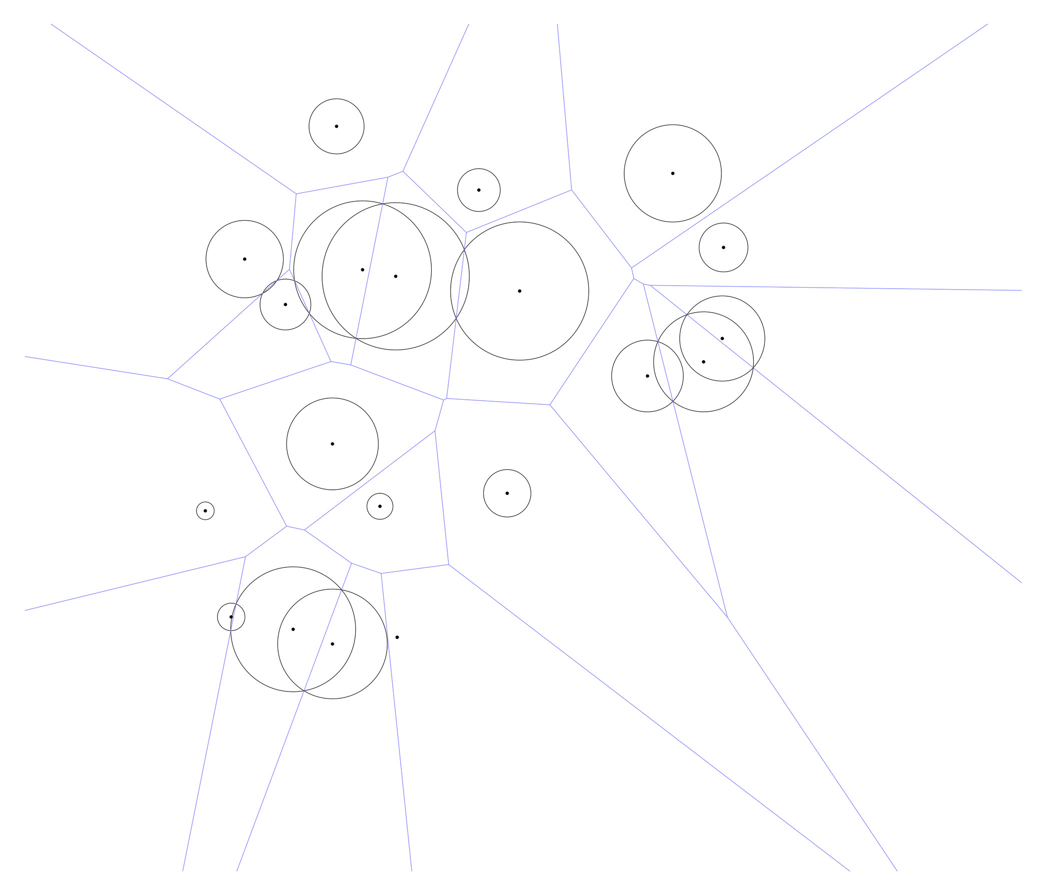

Figure 1: *The Laguerre Diagram of a set of balls in $\mathbb{R}^2$ is shown.*

The foundations for this work date back at least to Steiner, who coined the term "power" in this context.[3] Given two hyperballs B_1, B_2, the set

$$\{B_1 = B_2\} = \{x \mid \mathcal{P}(B_1, x) = \mathcal{P}(B_2, x)\}$$

is called the *radical axis* of the two hyperballs.[1] It is simple to show by projection that the radical axis of two non-concentric n-balls is an $(n-1)$-dimensional hyperplane perpendicular to the line through the centers of the two balls. In the two open halfspaces defined by the hyperplane, either of the two balls dominates the other. That is, one halfspace defined by the radical axis of B_1, B_2 is the set

$$\{B_1 \sqsubset B_2\} = \{x \mid \mathcal{P}(B_1, x) > \mathcal{P}(B_2, x)\} \subset \left\{x \mid x \in \overline{B_1} \implies x \in B_2\right\}$$

and the other halfspace is $\{B_2 \sqsubset B_1\}$. We define $\{B_1 \sqsubseteq B_2\}$ to be the closure of $\{B_1 \sqsubset B_2\}$, that is $\{B_1 \sqsubseteq B_2\} = \{x \mid \mathcal{P}(B_1, x) \geq \mathcal{P}(B_2, x)\}$. It is easy to see that the Laguerre cell may be recharacterized in terms of these dominant halfspaces, as follows:

$$\mathrm{Dom}_{\Omega}(\mathcal{B}) = \bigcap_{\mathcal{B}' \in \Omega} \{\mathcal{B}' \sqsubseteq \mathcal{B}\}.$$

It is also clear, then, that each Laguerre cell is the intersection of a finite number of closed halfspaces, and so is closed and convex.

Note that the power of a point x with respect to ball B may be negative and that

$$\mathcal{P}(B, x) = \begin{cases} < 0 \text{ iff } x \in B, \\ = 0 \text{ iff } x \in \partial B, \\ > 0 \text{ iff } x \notin \overline{B}. \end{cases}$$

It should be clear, then, that Laguerre Diagrams are relevant to the issue of *point location*.

Problem: Given a collection of n-balls, Ω, and a point $x \in \mathbb{R}^n$, find a $\mathcal{B} \in \Omega$ such that $x \in \mathcal{B}$, or report if there is none.

Solution: Find the $\mathcal{B} \in \Omega$ such that x is in the Laguerre cell, $\mathrm{Dom}_{\Omega}(\mathcal{B})$. If $x \in \mathcal{B}'$ for any $\mathcal{B}' \in \Omega$, then $x \in \mathcal{B}$. Thus it suffices to check if $x \in \mathcal{B}$.

First Steps

The following simple claims follow from the structure of $\mathbb{R}^n$.

Claim Given two nonconcentric n-dimensional hyperballs B_1, B_2, and a set of n affinely independent points, V contained in $\partial B_1 \cap \partial B_2$, then the radical axis, $\{B_1 = B_2\}$ is the hyperplane uniquely determined by the set V, since each point in V is in the radical axis.

Claim Given any two n-dimensional hyperballs B_1, B_2, with $\partial B_1 \cap \partial B_2 \neq \emptyset$, and a set of $n+1$ affinely independent vertices, $\{v_0, v_1, \ldots, v_n\}$, with $v_0 \in \partial B_1 \cap \partial B_2$, and $v_j \in \partial B_1 \setminus B_2$ for $j = 1, 2, \ldots, n$, then the positive cone of the vectors $\{v_j - v_0\}_{j=1}^{n}$ is contained in $\{B_2 \sqsubseteq B_1\}$. If the hyperballs are distinct, then the interior of the cone is contained in $\{B_2 \sqsubset B_1\}$.

Claim Let $V = \{v_i\}_{i=0}^{n+2}$ be a set of points in $\mathbb{R}^n$ such that the simplices S_1 on $\{v_i\}_{i=0}^{n+1}$ and S_2 on $\{v_i\}_{i=1}^{n+2}$ have the strong Delaunay property with respect to V. Then S_1 is contained in $\{\mathcal{C}(S_2) \sqsubseteq \mathcal{C}(S_1)\}$, and S_2 is in $\{\mathcal{C}(S_1) \sqsubseteq \mathcal{C}(S_2)\}$.

This fact allows us to prove the correctness and termination of the following well-known "walking" method for point location.

Problem: Given the Delaunay Triangulation of a set of points in $\mathbb{R}^n$, and a point x, find a triangle which x illuminates, *i.e.*, one whose circumball contains x, or report if there is none.

Solution: Start with some simplex s. If x illuminates s, then return s. Otherwise, x must be outside of s. Pick a neighboring simplex of s, s' such that x is on the opposite side of the hyperplane containing the face $s \cap s'$. If there is no such neighbor, then x does not illuminate any simplex. Otherwise, start over with the simplex s'.

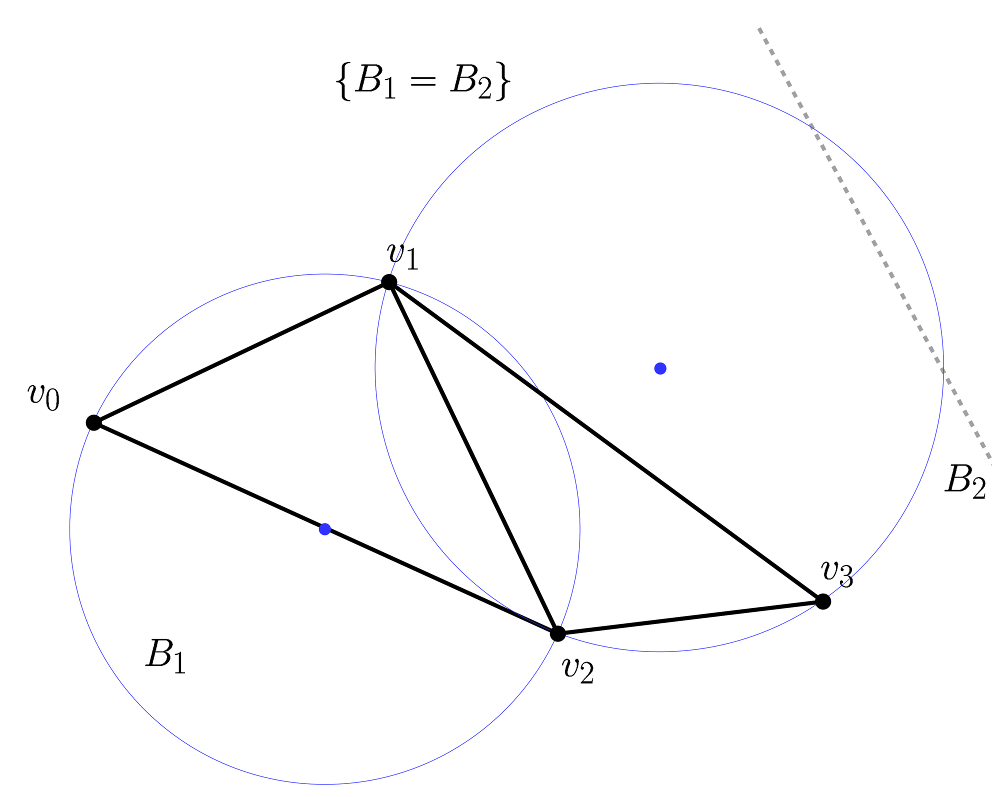

Figure 2: *The radical axis of two balls, B_1, B_2 is shown. The open halfspace to the upper right is $\{B_1 \sqsubset B_2\}$. Many of the claims concerning radical axis and dominating regions are illustrated in this figure,* e.g., *the positive cone of the vectors $\{v_0 - v_1, v_2 - v_1\}$ is contained in the set $\{B_2 \sqsubset B_1\}$; if S_1 is the triangle $\Delta v_0 v_1 v_2$, and S_2 is $\Delta v_1 v_2 v_3$, then $S_1 \subseteq \{\mathcal{C}(S_2) \sqsubseteq \mathcal{C}(S_1)\}$; If s is the segment (v_0, v_2), then the positive cylinder $\mathrm{Cyl}_G^{+}(s)$ is contained in $\{B_2 \sqsubseteq B_1\}$, where G is the line containing the segment, and "positive" is away from B_2.*

Claim Suppose B_1, B_2 are two n-dimensional hyperballs, such that the center of B_1 is in the $(n-1)$-dimensional hyperplane G. Furthermore assume that V is a set of l affinely independent vertices,

$$V = \{v_i\}_{i=0}^{i=l-1} \subset \partial B_1 \cap G,$$

for $2 \leq l \leq n$. Let S_{l-1} be the simplex defined by the vertices of V. Then

1. If the center of B_2 is on G, then $\{B_2 = B_1\}$ is a hyperplane perpendicular to G. Moreover $\mathrm{Cyl}_G(S_{l-1}) \subseteq \{B_2 \sqsubseteq B_1\}$, where $\mathrm{Cyl}_G(s)$ is the set of points whose projection onto G is contained in s.
2. If the center of B_2 is not on G then $\mathrm{Cyl}_G^{+}(S_{l-1}) \subseteq \{B_2 \sqsubseteq B_1\}$, where $\mathrm{Cyl}_G^{+}(s)$ is the set of points on a given side of G (in this case, opposite G from the center of B_2) whose projection onto G is contained in s.

2. Laguerre-Delaunay Duality

The main goal of this work is to simplify the task of proving various statements concerning Delaunay Triangulations. First we show that Laguerre Diagrams, like regular Voronoi Diagrams, are somehow independent of embedding. More specifically, by embedding the Laguerre sites in a higher dimension, the Laguerre cells become cylinders:

Claim Given a set of n-dimensional hyperballs, Ω, each of which has center located in the l-dimensional hyperplane, G, if $\Omega' = \{\mathcal{B} \cap G \mid \mathcal{B} \in \Omega\}$, then $\mathrm{Dom}_{\Omega}(\mathcal{B})$ is the set of points whose projection onto G is $\mathrm{Dom}_{\Omega'}(\mathcal{B} \cap G)$ for each $\mathcal{B} \in \Omega$.

The main result is as follows:

Claim Let $\mathbf{\Sigma}$ be a collection of n-simplices on vertices of V. Assume each simplex of $\mathbf{\Sigma}$ has the Delaunay Property with respect to V. Let $\Omega = \{\mathcal{C}(s) \mid s \in \mathbf{\Sigma}\}$. Then for every $s \in \mathbf{\Sigma}$,

$$s \subseteq \mathrm{Dom}_{\Omega}(\mathcal{C}(s)).$$

That is, a simplex is contained in the Laguerre cell of its circumball.

Let $K = \cup_{s \in \mathbf{\Sigma}} s$, and take $\mathbf{\Sigma}'$ to be the set of $(n-1)$-dimensional faces of simplices of $\mathbf{\Sigma}$ which are contained in the boundary of K. Suppose furthermore that for every $s' \in \mathbf{\Sigma}'$, that the circumball of s' does not contain any point of V. Then, letting $\Omega' = \left\{\mathcal{C}(s') \mid s' \in \mathbf{\Sigma}'\right\}$, for every $s \in \mathbf{\Sigma}$,

$$s = \mathrm{Dom}_{\Omega \cup \Omega'}(\mathcal{C}(s)).$$

That is, with regard to this augmented set of circumballs, every n-simplex is exactly equal to its Laguerre cell.

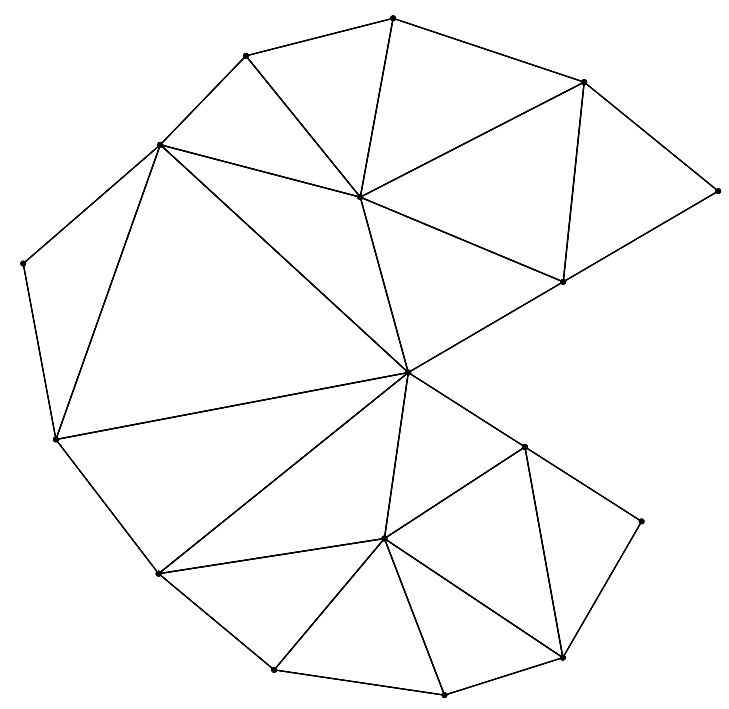

Figure 3: *A collection of simplices, $\mathbf{\Sigma}$, each with the Delaunay Property with respect to the set of vertices is shown. Each subsimplex on the boundary of K has a empty circumball, where K is the union of the simplices of $\mathbf{\Sigma}$.*

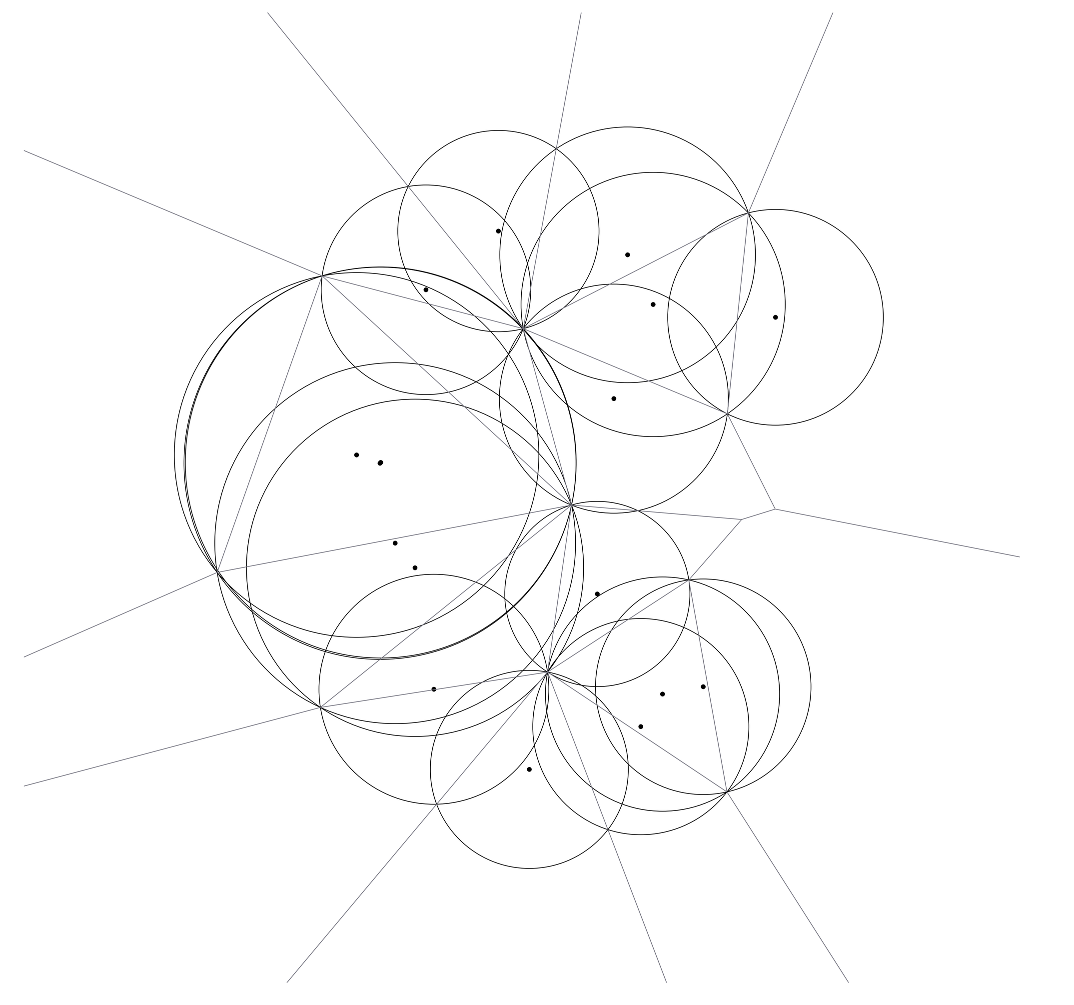

Figure 4: *The Laguerre Diagram of Ω, the set of circumballs of simplices of the set $\mathbf{\Sigma}$ from Figure 3., is shown. Each simplex of $\mathbf{\Sigma}$ is contained in its associated Laguerre cell.*

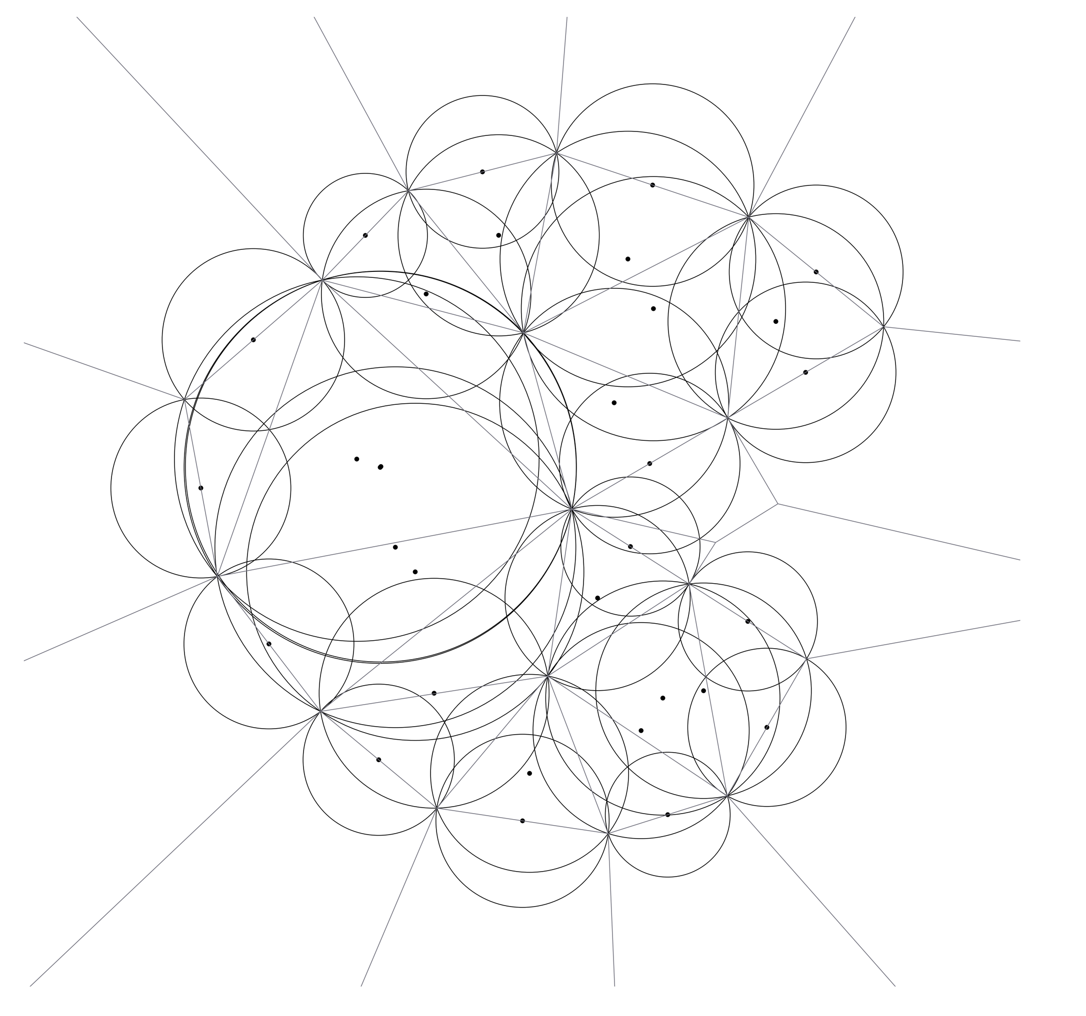

Figure 5: *The Laguerre Diagram of $\Omega \cup \Omega'$ is shown, where Ω' is the set of circumballs of the edge segments of $\mathbf{\Sigma}$, the set of simplices from Figure 3. Each simplex of $\mathbf{\Sigma}$ is now exactly its associated Laguerre cell.*

References

[1] H. S. M. Coxeter and S. L. Greitzer. *Geometry Revisited*. The Mathematical Association of America, Washington, DC, 1967. ISBN 0-88385-600-X.

[2] Atsuyuki Okabe, Barry Boots, Kokichi Sugihara and Sung Nok Chiu. *Spatial tessellations: concepts and applications of Voronoi diagrams*. John Wiley & Sons Ltd., Chichester, second edition, 2000. ISBN 0-471-98635-6. With a foreword by D. G. Kendall.

[3] Jacob Steiner. Einige geometrische betrachtungen. *J. reine angew. Math*, 1:161–184, 1826.

[4] Kokichi Sugihara. Open source software for computing Laguerre Diagrams, including code for 2-d, 3-d, and spherical Voronoi and Laguerre Diagrams, as well as convex hull in 2,3 and 4 dimensions. URL `http://www.simplex.t.u-tokyo.ac.jp/~sugihara/opensoft.`